

ON QUANTITATIVE OPERATOR K -THEORY

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ABSTRACT. In this paper, we develop a quantitative K -theory for filtered C^* -algebras. Particularly interesting examples of filtered C^* -algebras include group C^* -algebras, crossed product C^* -algebras and Roe algebras. We prove a quantitative version of the six term exact sequence and a quantitative Bott periodicity. We apply the quantitative K -theory to formulate a quantitative version of the Baum-Connes conjecture and prove that the quantitative Baum-Connes conjecture holds for a large class of groups.

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0. INTRODUCTION

The receptacles of higher indices of elliptic differential operators are K -theory of C^* -algebras which encode the (large scale) geometry of the underlying spaces. The following examples are important for purpose of applications to geometry and topology.

- K -theory of group C^* -algebras is a receptacle for higher index theory of equivariant elliptic differential operators on covering space [1, 2, 5, 11];
- K -theory of crossed product C^* -algebras and more generally groupoid C^* -algebras for foliations serve as receptacles for longitudinally elliptic operators [3, 4];
- the higher indices of elliptic operators on noncompact Riemannian manifolds live in K -theory of Roe algebras [15].

The local nature of differential operators implies that these higher indices can be defined in term of idempotents and invertible elements with finite propagation. Using homotopy invariance of the K -theory for C^* -algebras, these higher indices give rise to topological invariants.

In the context of Roe algebras, a quantitative operator K -theory was introduced to compute the higher indices of elliptic operators for noncompact spaces with finite asymptotic dimension [19]. The aim of this paper is to develop a quantitative K -theory for general C^* -algebras equipped with a filtration. The filtration structure allows us to define the concept of propagation. Examples of C^* -algebras with filtrations include group C^* -algebras, crossed product C^* -algebras and Roe algebras. The quantitative K -theory for C^* -algebras with filtrations is then defined in terms of homotopy of quasi-projections and quasi-unitaries with propagation and norm controls. We introduce controlled morphisms to study quantitative operator K -theory. In particular, we derive a quantitative version of the six term exact sequence. In the case of crossed product algebras, we also define a quantitative version of the Kasparov transformation compatible with Kasparov product. We end this paper by using the quantitative K -theory to formulate a quantitative version of the Baum-Connes conjecture and prove it for a large class of groups.

This paper is organized as follows: In section 1, we collect a few notations and definitions including the concept of filtered C^* -algebras. We use the concepts of almost unitary and almost projection to define a quantitative K -theory for filtered C^* -algebras and we study its elementary properties. In section 2, we introduce the notion of controlled morphism in quantitative K -theory. Section 3 is devoted to extensions of filtered C^* -algebras and to a controlled exact sequence for quantitative K -theory. In section 4, we prove a controlled version of the Bott periodicity and as a consequence, we obtain a controlled version of the six-term exact sequence in K -theory. In section 5, we apply KK -theory to study the quantitative K -theory of crossed product C^* -algebras and discuss its application to K -amenability. Finally

in section 8, we formulate a quantitative Baum-Connes conjecture and prove the quantitative Baum-Connes conjecture for a large class of groups.

1. QUANTITATIVE K -THEORY

In this section, we introduce a notion of quantitative K -theory for C^* -algebras with a filtration. Let us fix first some notations about C^* -algebras we shall use throughout this paper.

- If B is a C^* -algebra and if b_1, \dots, b_k are respectively elements of $M_{n_1}(B), \dots, M_{n_k}(B)$, we denote by $\text{diag}(b_1, \dots, b_k)$ the block diagonal matrix $\begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_k \end{pmatrix}$ of $M_{n_1+\dots+n_k}(B)$.
- If X is a locally compact space and B is a C^* -algebra, we denote by $C_0(X, B)$ the C^* -algebra of B -valued continuous functions on X vanishing at infinity. The special cases of $X = (0, 1]$, $X = [0, 1]$, $X = (0, 1)$ and $X = [0, 1]$, will be respectively denoted by CB , $B[0, 1]$, SB and $B[0, 1]$.
- For a separable Hilbert space \mathcal{H} , we denote by $\mathcal{K}(\mathcal{H})$ the C^* -algebra of compact operators on \mathcal{H} .
- If A and B are C^* -algebras, we will denote by $A \otimes B$ their spatial tensor product.

1.1. Filtered C^* -algebras.

Definition 1.1. A filtered C^* -algebra A is a C^* -algebra equipped with a family $(A_r)_{r>0}$ of linear subspaces indexed by positive numbers such that:

- $A_r \subset A_{r'}$ if $r \leq r'$;
- A_r is stable by involution;
- $A_r \cdot A_{r'} \subset A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in A .

If A is unital, we also require that the identity 1 is an element of A_r for every positive number r . The elements of A_r are said to have **propagation** r .

- Let A and A' be respectively C^* -algebras filtered by $(A_r)_{r>0}$ and $(A'_r)_{r>0}$. A homomorphism of C^* -algebras $\phi : A \rightarrow A'$ is a filtered homomorphism (or a homomorphism of filtered C^* -algebras) if $\phi(A_r) \subset A'_r$ for any positive number r .
- If A is a filtered C^* -algebra and X is a locally compact space, then $C_0(X, A)$ is a C^* -algebra filtered by $(C_0(X, A_r))_{r>0}$. In particular the algebras CA , $A[0, 1]$, $A[0, 1)$ and SA are filtered C^* -algebras.
- If A is a non unital filtered C^* -algebra, then its unitarization \tilde{A} is filtered by $(A_r + \mathbb{C})_{r>0}$. We define for A non-unital the homomorphism

$$\rho_A : \tilde{A} \rightarrow \mathbb{C}; a + z \mapsto z$$

for $a \in A$ and $z \in \mathbb{C}$.

Prominent examples of filtered C^* -algebra are provided by Roe algebras associated to proper metric spaces, i.e. metric spaces such that closed balls of given radius are compact. Recall that for such a metric space (X, d) , a X -module is a Hilbert space H_X together with a $*$ -representation ρ_X of $C_0(X)$ in H_X (we shall

write f instead of $\rho_X(f)$). If the representation is non-degenerate, the X -module is said to be non-degenerate. A X -module is called standard if no non-zero function of $C_0(X)$ acts as a compact operator on H_X .

The following concepts were introduced by Roe in his work on index theory of elliptic operators on noncompact spaces [15].

Definition 1.2. Let H_X be a standard non-degenerate X -module and let T be a bounded operator on H_X .

- (i) The support of T is the complement of the open subset of $X \times X$

$$\{(x, y) \in X \times X \text{ s.t. there exist } f \text{ and } g \text{ in } C_0(X) \text{ satisfying } f(x) \neq 0, g(y) \neq 0 \text{ and } f \cdot T \cdot g = 0\}.$$
- (ii) The operator T is said to have finite propagation (in this case propagation less than r) if there exists a real r such that for any x and y in X with $d(x, y) > r$, then (x, y) is not in the support of T .
- (iii) The operator T is said to be locally compact if $f \cdot T$ and $T \cdot f$ are compact for any f in $C_0(X)$. We then define $C[X]$ as the set of locally compact and finite propagation bounded operators of H_X , and for every $r > 0$, we define $C[X]_r$ as the set of element of $C[X]$ with propagation less than r .

We clearly have $C[X]_r \cdot C[X]_{r'} \subset C[X]_{r+r'}$. We can check that up to (non-canonical) isomorphism, $C[X]$ does not depend on the choice of H_X .

Definition 1.3. The Roe algebra $C^*(X)$ is the norm closure of $C[X]$ in the algebra $L(H_X)$ of bounded operators on H_X . The Roe algebra is then filtered by $(C[X]_r)_{r>0}$.

Although $C^*(X)$ is not canonically defined, it was proved in [9] that up to canonical isomorphisms, its K -theory does not depend on the choice of a non-degenerate standard X -module. Furthermore, $K_*(C^*(X))$ is the natural receptacle for higher indices of elliptic operators with support on X [15].

If X has bounded geometry, then the Roe algebra admits a maximal version [7] filtered by $(C[X]_r)_{r>0}$. Other important examples are reduced and maximal crossed product of a C^* -algebra by an action of a discrete group by automorphisms. These examples will be studied in detail in section 5.

1.2. Almost projections/unitaries. Let A be a unital filtered C^* -algebra. For any positive numbers r and ε , we call

- an element u in A a ε - r -unitary if u belongs to A_r , $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$. The set of ε - r -unitaries on A will be denoted by $U^{\varepsilon,r}(A)$.
- an element p in A a ε - r -projection if p belongs to A_r , $p = p^*$ and $\|p^2 - p\| < \varepsilon$. The set of ε - r -projections on A will be denoted by $P^{\varepsilon,r}(A)$.

For n integer, we set $U_n^{\varepsilon,r}(A) = U^{\varepsilon,r}(M_n(A))$ and $P_n^{\varepsilon,r}(A) = P^{\varepsilon,r}(M_n(A))$.

For any unital filtered C^* -algebra A , any positive numbers ε and r and any positive integer n , we consider inclusions

$$P_n^{\varepsilon,r}(A) \hookrightarrow P_{n+1}^{\varepsilon,r}(A); p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$U_n^{\varepsilon,r}(A) \hookrightarrow U_{n+1}^{\varepsilon,r}(A); u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

This allows us to define

$$U_{\infty}^{\varepsilon,r}(A) = \bigcup_{n \in \mathbb{N}} U_n^{\varepsilon,r}(A)$$

and

$$P_{\infty}^{\varepsilon,r}(A) = \bigcup_{n \in \mathbb{N}} P_n^{\varepsilon,r}(A).$$

Remark 1.4. Let r and ε be positive numbers with $\varepsilon < 1/4$;

- (i) If p is an ε - r -projection in A , then the spectrum of p is included in $\left(\frac{1-\sqrt{1-4\varepsilon}}{2}, \frac{1+\sqrt{1-4\varepsilon}}{2}\right) \cup \left(\frac{1+\sqrt{1-4\varepsilon}}{2}, \frac{1+\sqrt{1+4\varepsilon}}{2}\right)$ and thus $\|p\| < 1 + \varepsilon$.
- (ii) If u is an ε - r -unitary in A , then

$$1 - \varepsilon < \|u\| < 1 + \varepsilon/2,$$

$$1 - \varepsilon/2 < \|u^{-1}\| < 1 + \varepsilon,$$

$$\|u^* - u^{-1}\| < (1 + \varepsilon)\varepsilon.$$

- (iii) Let $\kappa_{0,\varepsilon} : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $\kappa_{0,\varepsilon}(t) = 0$ if $t \leq \frac{1-\sqrt{1-4\varepsilon}}{2}$ and $\kappa_{0,\varepsilon}(t) = 1$ if $t \geq \frac{1+\sqrt{1-4\varepsilon}}{2}$. If p is an ε - r -projection in A , then $\kappa_{0,\varepsilon}(p)$ is a projection such that $\|p - \kappa_{0,\varepsilon}(p)\| < 2\varepsilon$ which moreover does not depend on the choice of $\kappa_{0,\varepsilon}$. From now on, we shall denote this projection by $\kappa_0(p)$.
- (iv) If u is an ε - r -unitary in A , set $\kappa_1(u) = u(u^*u)^{-1/2}$. Then $\kappa_1(u)$ is a unitary such that $\|u - \kappa_1(u)\| < \varepsilon$.
- (v) If p is an ε - r -projection in A and q is a projection in A such that $\|p - q\| < 1 - 2\varepsilon$, then $\kappa_0(p)$ and q are homotopic projections [18, Chapter 5].
- (vi) If u and v are ε - r -unitaries in A , then uv is an $\varepsilon(2 + \varepsilon)$ - $2r$ -unitary in A .

Definition 1.5. Let A be a C^* -algebra filtered by $(A_r)_{r>0}$.

- Let p_0 and p_1 be ε - r -projections. We say that p_0 and p_1 are homotopic ε - r -projections if there exists a ε - r -projection q in $A[0, 1]$ such that $q(0) = p_0$ and $q(1) = p_1$. In this case, q is called a homotopy of ε - r -projections in A and will be denoted by $(q_t)_{t \in [0,1]}$.
- If A is unital, let u_0 and u_1 be ε - r -unitaries. We say that u_0 and u_1 are homotopic ε - r -unitaries if there exists an ε - r -unitary v in $A[0, 1]$ such that $v(0) = u_0$ and $v(1) = u_1$. In this case, v is called a homotopy of ε - r -unitaries in A and will be denoted by $(v_t)_{t \in [0,1]}$.

Example 1.6. Let p be a ε -projection in a filtered unital C^* -algebra A . Set $c_t = \cos \pi t/2$ and $s_t = \sin \pi t/2$ for $t \in [0, 1]$ and let us consider the homotopy of projections $(h_t)_{t \in [0,1]}$ with $h_t = \begin{pmatrix} c_t^2 & c_t s_t \\ c_t s_t & s_t^2 \end{pmatrix}$ in $M_2(\mathbb{C})$ between $\text{diag}(1, 0)$ and $\text{diag}(0, 1)$. Set $(q_t)_{t \in [0,1]} = (\text{diag}(p, 0) + (1-p) \otimes h_t)_{t \in [0,1]}$. Since $q_t^2 - q_t = s_t^2(p^2 - p) \otimes I_2$, we see that $(q_t)_{t \in [0,1]}$ is a homotopy of ε - r -projections between $\text{diag}(1, 0)$ and $\text{diag}(p, 1-p)$ in $M_2(A)$.

Next result will be frequently used throughout the paper and is quite easy to prove.

Lemma 1.7. Let A be a C^* -algebra filtered by $(A_r)_{r>0}$.

- (i) If p is an ε - r -projection in A and q is a self-adjoint element of A_r such that $\|p - q\| < \frac{\varepsilon - \|p^2 - p\|}{4}$, then q is ε - r -projection. In particular, if p is an ε - r -projection in A and if q is a self-adjoint element in A_r such that $\|p - q\| < \varepsilon$, then q is a 5ε - r -projection in A and p and q are connected by a homotopy of 5ε - r -projections.
- (ii) If A is unital and if u is an ε - r -unitary and v is an element of A_r such that $\|u - v\| < \frac{\varepsilon - \|u^*u - 1\|}{3}$, then v is an ε - r -unitary. In particular, if u is an ε - r -unitary and v is an element of A_r such that $\|u - v\| < \varepsilon$, then v is an 4ε - r -unitary in A and u and v are connected by a homotopy of 4ε - r -unitaries.

Lemma 1.8. *There exists a real $\lambda > 4$ such that for any positive number ε with $\varepsilon < 1/\lambda$, any positive real r , any ε - r -projection p and ε - r -unitary W in a filtered unital C^* -algebra A , the following assertions hold:*

- (i) WpW^* is a $\lambda\varepsilon$ - $3r$ -projection of A ;
- (ii) $\text{diag}(WpW^*, 1)$ and $\text{diag}(p, 1)$ are homotopic $\lambda\varepsilon$ - $3r$ -projections.

Proof. The first point is straightforward to check from remark 1.4. For the second point, with notations of example 1.6, use the homotopy of ε - r -unitaries

$$\left(\begin{array}{cc} Wc_t^2 + s_t^2 & (W-1)s_t c_t \\ (W-1)s_t c_t & Ws_t^2 + c_t^2 \end{array} \right)_{t \in [0,1]} = \left(\begin{pmatrix} c_t & -s_t \\ s_t & c_t \end{pmatrix} \cdot \text{diag}(W, 1) \cdot \begin{pmatrix} c_t & s_t \\ -s_t & c_t \end{pmatrix} \right)_{t \in [0,1]}$$

to connect by conjugation $\text{diag}(WpW^*, 1)$ to $\text{diag}(p, WW^*)$ and then connect to $\text{diag}(p, 1)$ by a ray. \square

Recall that if two projections in a unital C^* -algebra are close enough in norm, then there are conjugated by a canonical unitary. To state a similar result in term of ε - r -projections and ε - r -unitaries, we will need the definition of a control pair.

Definition 1.9. *A control pair is a pair (λ, h) , where*

- $\lambda > 1$;
- $h : (0, \frac{1}{4\lambda}) \rightarrow (0, +\infty)$; $\varepsilon \mapsto h_\varepsilon$ is a map such that there exists a non-increasing map $g : (0, \frac{1}{4\lambda}) \rightarrow (0, +\infty)$, with $h \leq g$.

Lemma 1.10. *There exists a control pair (λ, h) such that the following holds:*

for every positive number r , any ε in $(0, \frac{1}{4\lambda})$ and any ε - r -projections p and q of a filtered unital C^ -algebra A satisfying $\|p - q\| < 1/16$, there exists an $\lambda\varepsilon$ - $h_\varepsilon r$ -unitary W in A such that $\|WpW^* - q\| \leq \lambda\varepsilon$.*

Proof. We follow the proof of [18, Proposition 5.2.6]. If we set

$$z = (2\kappa_0(p) - 1)(2\kappa_0(q) - 1) + 1,$$

- then

$$\begin{aligned} \|z - 2\| &\leq 2\|\kappa_0(p) - \kappa_0(q)\| \\ &\leq 8\varepsilon + 2\|p - q\| \end{aligned}$$

and hence z is invertible for $\varepsilon < 1/16$.

- Moreover, if we set $U = z|z^{-1}|$ and since $z\kappa_0(q) = \kappa_0(p)z$, then we have $\kappa_0(q) = U\kappa_0(p)U^*$.

Let us define $z' = (2p - 1)(2q - 1) + 1$. Then we have $\|z - z'\| \leq 9\varepsilon$ and $\|z'\| \leq 3$. If ε is small enough, then $\|z'^*z' - 4\| \leq 2$ and hence the spectrum of z'^*z' is in $[2, 6]$. Let us consider the expansion in power series $\sum_{k \in \mathbb{N}} a_k t^k$ of $t \mapsto (1 + t)^{-1/2}$ on $(0, 1)$ and let n_ε be the smallest integer such that $\sum_{n_\varepsilon \leq k} |a_k|/2^k \leq \varepsilon$. Let us set then $W = z'/2 \sum_{k=0}^{n_\varepsilon} a_k (\frac{z'^*z'-4}{4})^k$. Then for a suitable λ (not depending on A, p, q or ε), we get that W is a $\lambda\varepsilon(4n_\varepsilon + 2)r$ -unitary which satisfies the required condition. \square

Remark 1.11. *The order of h when ε goes to zero in lemma 1.10 is $C\varepsilon^{-3/2}$ for some constant C .*

1.3. Definition of quantitative K -theory. For a unital filtered C^* -algebra A , we define the following equivalence relations on $P_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and on $U_\infty^{\varepsilon,r}(A)$:

- if p and q are elements of $P_\infty^{\varepsilon,r}(A)$, l and l' are positive integers, $(p, l) \sim (q, l')$ if there exists a positive integer k and an element h of $P_\infty^{\varepsilon,r}(A[0, 1])$ such that $h(0) = \text{diag}(p, I_{k+l'})$ and $h(1) = \text{diag}(q, I_{k+l})$.
- if u and v are elements of $U_\infty^{\varepsilon,r}(A)$, $u \sim v$ if there exists an element h of $U_\infty^{\varepsilon,r}(A[0, 1])$ such that $h(0) = u$ and $h(1) = v$.

If p is an element of $P_\infty^{\varepsilon,r}(A)$ and l is an integer, we denote by $[p, l]_{\varepsilon,r}$ the equivalence class of (p, l) modulo \sim and if u is an element of $U_\infty^{\varepsilon,r}(A)$ we denote by $[u]_{\varepsilon,r}$ its equivalence class modulo \sim .

Definition 1.12. *Let r and ε be positive numbers with $\varepsilon < 1/4$. We define:*

- (i) $K_0^{\varepsilon,r}(A) = P_\infty^{\varepsilon,r}(A) \times \mathbb{N} / \sim$ for A unital and
 $K_0^{\varepsilon,r}(A) = \{[p, l]_{\varepsilon,r} \in P_\infty^{\varepsilon,r}(\tilde{A}) \times \mathbb{N} / \sim \text{ such that } \dim \kappa_0(\rho_A(p)) = l\}$
for A non unital.
- (ii) $K_1^{\varepsilon,r}(A) = U_\infty^{\varepsilon,r}(\tilde{A}) / \sim$ (with $A = \tilde{A}$ if A is already unital).

Remark 1.13. *We shall see in lemma 1.24 that as it is the case for K -theory, $K_*^{\varepsilon,r}(\bullet)$ can indeed be defined in a uniform way for unital and non-unital filtered C^* -algebras.*

It is straightforward to check that for any unital filtered C^* -algebra A , if p is an ε - r -projection in A and u is an ε - r -unitary in A , then $\text{diag}(p, 0)$ and $\text{diag}(0, p)$ are homotopic ε - r -projections in $M_2(A)$ and $\text{diag}(u, 1)$ and $\text{diag}(1, u)$ are homotopic ε - r -unitaries in $M_2(A)$. Thus we obtain the following:

Lemma 1.14. *Let A be a filtered C^* -algebra. Then $K_0^{\varepsilon,r}(A)$ and $K_1^{\varepsilon,r}(A)$ are equipped with a structure of abelian semi-group such that*

$$[p, l]_{\varepsilon,r} + [p', l']_{\varepsilon,r} = [\text{diag}(p, p'), l + l']_{\varepsilon,r}$$

and

$$[u]_{\varepsilon,r} + [u']_{\varepsilon,r} = [\text{diag}(u, v)]_{\varepsilon,r},$$

for any $[p, l]_{\varepsilon,r}$ and $[p', l']_{\varepsilon,r}$ in $K_0^{\varepsilon,r}(A)$ and any $[u]_{\varepsilon,r}$ and $[u']_{\varepsilon,r}$ in $K_1^{\varepsilon,r}(A)$.

According to example 1.6, for every unital filtered C^* -algebra A , any ε - r -projection p in $M_n(A)$ and any integer l with $n \geq l$, we see that $[I_n - p, n - l]_{\varepsilon,r}$ is an inverse for $[p, l]_{\varepsilon,r}$. Hence we obtain:

Lemma 1.15. *If A is a filtered C^* -algebra, then $K_0^{\varepsilon,r}(A)$ is an abelian group.*

Although $K_1^{\varepsilon,r}(A)$ is not a group, it is very close to be one.

Lemma 1.16. *Let A be a filtered C^* -algebra. Then for any ε - r -unitary u in $M_n(\tilde{A})$ (with $A = \tilde{A}$ if A is already unital), we have $[u]_{3\varepsilon, 2r} + [u^*]_{3\varepsilon, 2r} = 0$ in $K_1^{3\varepsilon, 2r}(A)$.*

Proof. If u is an ε - r -unitary in a unital filtered C^* -algebra A , then according to point (vi) of remark 1.4, we see that $(\text{diag}(1, u) \begin{pmatrix} c_t & -s_t \\ s_t & c_t \end{pmatrix} \cdot \text{diag}(1, u^*) \cdot \begin{pmatrix} c_t & s_t \\ -s_t & c_t \end{pmatrix})_{t \in [0, 1]}$ is a homotopy of 3ε - $2r$ -unitaries between $\text{diag}(u, u^*)$ and $\text{diag}(uu^*, 1)$. Since $\|uu^* - 1\| < \varepsilon$, we deduce from lemma 1.7 that uu^* and 1 are homotopic 3ε - $2r$ -unitaries and hence we get the lemma. \square

Remark 1.17. *According to lemma 1.16, if we define the equivalence relation on $U_\infty^{\varepsilon, r}(A)$ to be homotopy within $U_\infty^{3\varepsilon, 2r}(A)$, then $K_1^{\varepsilon, r}(A)$ can be endowed with an abelian group structure.*

We have for any filtered C^* -algebra A and any positive numbers r, r', ε and ε' with $\varepsilon \leq \varepsilon' < 1/4$ and $r \leq r'$ natural semi-group homomorphisms

- $\iota_0^{\varepsilon, r} : K_0^{\varepsilon, r}(A) \longrightarrow K_0(A); [p, l]_{\varepsilon, r} \mapsto [\kappa_0(p)] - [l];$
- $\iota_1^{\varepsilon, r} : K_1^{\varepsilon, r}(A) \longrightarrow K_1(A); [u]_{\varepsilon, r} \mapsto [u];$
- $\iota_*^{\varepsilon, r} = \iota_0^{\varepsilon, r} \oplus \iota_1^{\varepsilon, r};$
- $\iota_0^{\varepsilon, \varepsilon', r, r'} : K_0^{\varepsilon, r}(A) \longrightarrow K_0^{\varepsilon', r'}(A); [p, l]_{\varepsilon, r} \mapsto [p, l]_{\varepsilon', r'};$
- $\iota_1^{\varepsilon, \varepsilon', r, r'} : K_1^{\varepsilon, r}(A) \longrightarrow K_1^{\varepsilon', r'}(A); [u]_{\varepsilon, r} \mapsto [u]_{\varepsilon', r'}.$
- $\iota_*^{\varepsilon, \varepsilon', r, r'} = \iota_0^{\varepsilon, \varepsilon', r, r'} \oplus \iota_1^{\varepsilon, \varepsilon', r, r'}.$

If some of the indices r, r' or $\varepsilon, \varepsilon'$ are equal, we shall not repeat it in $\iota_*^{\varepsilon, \varepsilon', r, r'}$.

Remark 1.18. *Let p_0 and p_1 be two ε - r -projections in a filtered C^* -algebra such that $\kappa_0(p_0)$ and $\kappa_0(p_1)$ are homotopic projections. Then for any ε in $(0, 1/4)$, this homotopy can be approximated for some r' by a ε - r' -projection. Hence, using point (iii) of remark 1.4, there exists a homotopy $(q_t)_{t \in [0, 1]}$ of ε - r' projections in A such that $\|p_0 - q_0\| < 3\varepsilon$ and $\|p_1 - q_1\| < 3\varepsilon$. We can indeed assume that $r' \geq r$ and thus by lemma 1.7, we get that p_0 and p_1 are homotopic as 15ε - r' -projections. Proceeding in the same way for the odd case we eventually obtain:*

there exists $\lambda > 1$ such that for any filtered C^ -algebra A , any $\varepsilon \in (0, \frac{1}{4\lambda})$ and any positive number r , the following holds:*

Let x and x' be elements in $K_^{\varepsilon, r}(A)$ such that $\iota_*^{\varepsilon, r}(x) = \iota_*^{\varepsilon, r}(x')$ in $K_*(A)$, then there exists a positive number r' with $r' > r$ such that $\iota_*^{\varepsilon, \lambda\varepsilon, r, r'}(x) = \iota_*^{\varepsilon, \lambda\varepsilon, r, r'}(x')$ in $K_*^{\lambda\varepsilon, r'}(A)$.*

Lemma 1.19. *Let p be a matrix in $M_n(\mathbb{C})$ such that $p = p^*$ and $\|p^2 - p\| < \varepsilon$ for some ε in $(0, 1/4)$. Then there is a continuous path $(p_t)_{t \in [0, 1]}$ in $M_n(\mathbb{C})$ such that*

- $p_0 = p;$
- $p_1 = I_k$ with $k = \dim \kappa_0(p);$
- $p_t^* = p_t$ and $\|p_t^2 - p_t\| < \varepsilon$ for every t in $[0, 1].$

Proof. The selfadjoint matrix p satisfies $\|p^2 - p\| < \varepsilon$ if and only if the eigenvalues of p satisfy the inequality

$$-\varepsilon < \lambda^2 - \lambda < \varepsilon,$$

i.e.

$$\lambda \in \left(\frac{1 - \sqrt{1 + 4\varepsilon}}{2}, \frac{1 + \sqrt{1 - 4\varepsilon}}{2} \right) \cup \left(\frac{\sqrt{1 - 4\varepsilon} + 1}{2}, \frac{\sqrt{1 + 4\varepsilon} + 1}{2} \right).$$

Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of p lying in $\left(\frac{1-\sqrt{1+4\varepsilon}}{2}, \frac{1-\sqrt{1-4\varepsilon}}{2}\right)$ and let $\lambda_{k+1}, \dots, \lambda_n$ be the eigenvalues of p lying in $\left(\frac{\sqrt{1-4\varepsilon}+1}{2}, \frac{\sqrt{1+4\varepsilon}+1}{2}\right)$. We set for $t \in [0, 1]$

- $\lambda_{i,t} = t\lambda_i$ for $i = 1, \dots, k$;
- $\lambda_{i,t} = t\lambda_i + 1 - t$ for $i = k+1, \dots, n$.

Since $\lambda \mapsto \lambda^2 - \lambda$ is decreasing on $\left(\frac{1-\sqrt{1+4\varepsilon}}{2}, \frac{1-\sqrt{1-4\varepsilon}}{2}\right)$ and increasing on $\left(\frac{\sqrt{1-4\varepsilon}+1}{2}, \frac{\sqrt{1+4\varepsilon}+1}{2}\right)$ then,

$$-\varepsilon < \lambda_{i,t}^2 - \lambda_{i,t} < \varepsilon$$

for all t in $[0, 1]$ and $i = 1, \dots, n$. If we set $p_t = u \cdot \text{diag}(\lambda_{1,t}, \dots, \lambda_{n,t}) \cdot u^*$ where u is a unitary matrix of $M_n(\mathbb{C})$ such that $p = u \cdot \text{diag}(\lambda_1, \dots, \lambda_n) \cdot u^*$, then

- $p_0 = p$;
- $p_1 = \kappa_0(p)$;
- $p_t^* = p_t$ and $\|p_t^2 - p_t\| < \varepsilon$ for every t in $[0, 1]$.

Since there is a homotopy of projections in $M_n(\mathbb{C})$ between $\kappa_0(p)$ and I_k with $k = \dim \kappa_0(p)$, we get the result. \square

As a consequence we obtain:

Corollary 1.20. *For any positive numbers r and ε with $\varepsilon < 1/4$, then*

$$K_0^{\varepsilon,r}(\mathbb{C}) \rightarrow \mathbb{Z}; [p, l]_{\varepsilon,r} \mapsto \dim \kappa_0(p) - l$$

is an isomorphism.

Lemma 1.21. *Let u be a matrix in $M_n(\mathbb{C})$ such that $\|u^*u - I_n\| < \varepsilon$ and $\|uu^* - I_n\| < \varepsilon$ for ε in $(0, 1/4)$. Then there is a continuous path $(u_t)_{t \in [0,1]}$ in $M_n(\mathbb{C})$ such that*

- $u_0 = u$;
- $u_1 = I_n$;
- $\|u_t^*u_t - I_n\| < \varepsilon$ and $\|u_t u_t^* - I_n\| < \varepsilon$ for every t in $[0, 1]$.

Proof. Since u is invertible, u^*u and uu^* have the same eigenvalues $\lambda_1, \dots, \lambda_n$, and thus $\|u_t^*u_t - I_n\| < \varepsilon$ and $\|u_t u_t^* - I_n\| < \varepsilon$ if and only if $\lambda_i \in (1 - \varepsilon, 1 + \varepsilon)$ for $i = 1, \dots, n$. Let us set

- $h_t = w \cdot \text{diag}(\lambda_1^{-t/2}, \dots, \lambda_n^{-t/2}) \cdot w^*$ where w is a unitary matrix of $M_n(\mathbb{C})$ such that $u^*u = w \cdot \text{diag}(\lambda_1, \dots, \lambda_n) \cdot w^*$;
- $v_t = u \cdot h_t$ for all $t \in [0, 1]$. Then $v_t^*v_t = w \cdot \text{diag}(\lambda_1^{1-t}, \dots, \lambda_n^{1-t}) \cdot w^*$.

Since $|\lambda_i^{1-t} - 1| < \varepsilon$ for all $t \in [0, 1]$, we get that $\|v_t^*v_t - I_n\| < \varepsilon$ and $\|v_t v_t^* - I_n\| < \varepsilon$ for every t in $[0, 1]$. The matrix v_1 is unitary and the result then follows from path-connectness of $U_n(\mathbb{C})$. \square

As a consequence we obtain:

Corollary 1.22. *For any positive numbers r and ε with $\varepsilon < 1/4$, then we have*
 $K_1^{\varepsilon,r}(\mathbb{C}) = \{0\}$.

1.4. Elementary properties of quantitative K -theory. Let A_1 and A_2 be two unital C^* -algebras respectively filtered by $(A_{1,r})_{r>0}$ and $(A_{2,r})_{r>0}$ and consider $A_1 \oplus A_2$ filtered by $(A_{1,r} \oplus A_{2,r})_{r>0}$. Since we have identifications $P_\infty^{\varepsilon,r}(A_1 \oplus A_2) \cong P_\infty^{\varepsilon,r}(A_1) \times P_\infty^{\varepsilon,r}(A_2)$ and $U_\infty^{\varepsilon,r}(A_1 \oplus A_2) \cong U_\infty^{\varepsilon,r}(A_1) \times U_\infty^{\varepsilon,r}(A_2)$ induced by the inclusions $A_1 \hookrightarrow A_1 \oplus A_2$ and $A_2 \hookrightarrow A_1 \oplus A_2$, we see that we have isomorphisms $K_0^{\varepsilon,r}(A_1) \oplus K_0^{\varepsilon,r}(A_2) \xrightarrow{\sim} K_0^{\varepsilon,r}(A_1 \oplus A_2)$ and $K_1^{\varepsilon,r}(A_1) \oplus K_1^{\varepsilon,r}(A_2) \xrightarrow{\sim} K_1^{\varepsilon,r}(A_1 \oplus A_2)$.

Lemma 1.23. *Let A be a filtered non unital C^* -algebra and let ε and r be positive numbers with $\varepsilon < 1/4$. We have a natural splitting*

$$K_0^{\varepsilon,r}(\tilde{A}) \xrightarrow{\cong} K_0^{\varepsilon,r}(A) \oplus \mathbb{Z}.$$

Proof. Viewing A as a subalgebra of \tilde{A} , the group homomorphisms

$$\begin{aligned} K_0^{\varepsilon,r}(\tilde{A}) &\longrightarrow K_0^{\varepsilon,r}(A) \oplus \mathbb{Z} \\ [p, l]_{\varepsilon,r} &\mapsto ([p, \dim \kappa_0(\rho_A(p))]_{\varepsilon,r}, \dim \kappa_0(\rho_A(p)) - l) \end{aligned}$$

and

$$\begin{aligned} K_0^{\varepsilon,r}(A) \oplus \mathbb{Z} &\longrightarrow K_0^{\varepsilon,r}(\tilde{A}) \\ ([p, l]_{\varepsilon,r}, k - k') &\mapsto \left[\begin{pmatrix} p & 0 \\ 0 & I_k \end{pmatrix}, l + k' \right]_{\varepsilon,r} \end{aligned}$$

are inverse one of the other. \square

Let us set $A^+ = A \oplus \mathbb{C}$ equipped with the multiplication

$$(a, x) \cdot (b, y) = (ab + xb + ya, xy)$$

for a and b in A and x and y in \mathbb{C} . Notice that

- A^+ is isomorphic to $A \oplus \mathbb{C}$ with the algebra structure provided by the direct sum if A is unital;
- $A^+ = \tilde{A}$ if A is not unital.

Let us define also ρ_A in the unital case by $\rho_A : A^+ \rightarrow \mathbb{C}; (a, x) \mapsto x$. We know that in usual K -theory, we can equivalently define for A unital the \mathbb{Z}_2 -graded group $K_*(A)$ as A^+ by

$$K_0(A) = \ker \rho_{A,*} : K_0(A^+) \rightarrow K_0(\mathbb{C}) \cong \mathbb{Z}$$

and

$$K_1(A) = K_1(A^+).$$

Let us check that this is also the case for our \mathbb{Z}_2 -graded semi-groups $K_*^{\varepsilon,r}(A)$. If the C^* -algebra A is filtered by $(A_r)_{r>0}$, then A^+ is filtered by $(A_r + \mathbb{C})_{r>0}$. Let us define for a unital filtered algebra A

$$K_0'^{\varepsilon,r}(A) = \{[p, l]_{\varepsilon,r} \in P_\infty^{\varepsilon,r}(A^+) \times \mathbb{N} / \sim \text{ such that } \dim \kappa_0(\rho_A(p)) = l\}$$

and

$$K_1'^{\varepsilon,r}(A) = U_\infty^{\varepsilon,r}(A^+) / \sim.$$

Proceeding as we did in the proof of lemma 1.23, we obtain a natural splitting

$$K_0^{\varepsilon,r}(A^+) \xrightarrow{\cong} K_0'^{\varepsilon,r}(A) \oplus \mathbb{Z}.$$

But then, using the identification $A^+ \cong A \oplus \mathbb{C}$ and in view of lemmas 1.19 and 1.21, we get

Lemma 1.24. *The \mathbb{Z}_2 -graded semi-groups $K_*^{\varepsilon,r}(A)$ and $K_*'^{\varepsilon,r}(A)$ are naturally isomorphic.*

This allows us to state functoriality properties for quantitative K -theory. If $\phi : A \rightarrow B$ is a homomorphism of unital filtered C^* -algebras, then since ϕ preserve ε - r -projections and ε - r -unitaries, it obviously induces for any positive number r and any $\varepsilon \in (0, 1/4)$ a semi-group homomorphism

$$\phi_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon,r}(B).$$

In the non unital case, we can extend any homomorphism $\phi : A \rightarrow B$ to a homomorphism $\phi^+ : A^+ \rightarrow B^+$ of unital filtered C^* -algebras and then we use lemmas 1.23 and 1.24 to define $\phi_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon,r}(B)$. Hence, for any positive number r and any $\varepsilon \in (0, 1/4)$, we get that $K_0^{\varepsilon,r}(\bullet)$ (resp. $K_1^{\varepsilon,r}(\bullet)$) is a covariant additive functor from the category of filtered C^* -algebras (together with filtered homomorphism) to the category of abelian groups (resp. semi-groups).

Definition 1.25.

- (i) *Let A and B be filtered C^* -algebras. Then two homomorphisms of filtered C^* -algebras $\psi_0 : A \rightarrow B$ and $\psi_1 : A \rightarrow B$ are homotopic if there exists a path of homomorphisms of filtered C^* -algebras $\psi_t : A \rightarrow B$ for $0 \leq t \leq 1$ between ψ_0 and ψ_1 and such that $t \mapsto \psi_t$ is continuous for the pointwise norm convergence.*
- (ii) *A filtered C^* -algebra A is said to be contractible if the identity map and the zero map of A are homotopic.*

Example 1.26. *If A is a filtered C^* -algebra A , then the cone of A*

$$CA = \{f \in C([0, 1], A) \text{ such that } f(0) = 0\}$$

is a contractible filtered C^ -algebra.*

We have then the following obvious result:

Lemma 1.27. *If $\phi : A \rightarrow B$ and $\phi' : A \rightarrow B$ are two homotopic homomorphisms of filtered C^* -algebras, then $\phi_*^{\varepsilon,r} = \phi'^{\varepsilon,r}_*$ for every positive numbers ε and r with $\varepsilon < 1/4$. In particular, if A is a contractible filtered C^* -algebra, then $K_*^{\varepsilon,r}(A) = 0$ for every positive numbers ε and r with $\varepsilon < 1/4$.*

Let A be a C^* -algebra filtered by $(A_r)_{r>0}$ and let $(B_k)_{k \in \mathbb{N}}$ be an increasing sequence of C^* -subalgebras of A such that $\bigcup_{k \in \mathbb{N}} B_k$ is dense in A . Assume that $\bigcup_{r>0} B_k \cap A_r$ is dense in B_k for every integer k . Then for every integer k , the C^* -algebra B_k is filtered by $(B_k \cap A_r)_{r>0}$. If A is unital, then B_k is unital for some k , and thus we will assume without loss of generality that B_k is unital for every integer k .

Proposition 1.28. *Let A be a unital C^* -algebra filtered by $(A_r)_{r>0}$ and let $(B_k)_{k \in \mathbb{N}}$ be an increasing sequence of C^* -subalgebra of A such that*

- $\bigcup_{r>0} (B_k \cap A_r)$ is dense in B_k for every integer k ,
- $\bigcup_{k \in \mathbb{N}} (B_k \cap A_r)$ is dense in A_r for every positive number r .

Then the \mathbb{Z}_2 -graded semi-groups $K_^{\varepsilon,r}(A)$ and $\lim_k K_*^{\varepsilon,r}(B_k)$ are isomorphic.*

Proof. In particular, we see that $\bigcup_{k \in \mathbb{N}} B_k$ is dense in A . Let us denote by

$$\Upsilon_{*,\varepsilon,r} : \lim_k K_*^{\varepsilon,r}(B_k) \rightarrow K_*^{\varepsilon,r}(A)$$

the homomorphism of semi-group induced by the family of inclusions $B_k \hookrightarrow A$ where k runs through integers. We give the proof in the even case, the odd case being analogous. Let p be an element of $P_n^{\varepsilon,r}(A)$ and let $\delta = \|p^2 - p\| > 0$ and choose $\alpha < \frac{\varepsilon - \delta}{12}$. Since $\bigcup_{k \in \mathbb{N}} (B_k \cap A_r)$ is dense in A_r , there is an integer k and a selfadjoint element q of $M_n(B_k \cap A_r)$ such that $\|p - q\| < \alpha$. According to lemma 1.19, q is a ε - r projection. Let q' be another selfadjoint element of $M_n(B_k \cap A_r)$ such that $\|p - q'\| < \alpha$. Then $\|q - q'\| < 2\alpha$ and if we set $q_t = (1 - t)q + tq'$ for $t \in [0, 1]$, then

$$\begin{aligned} \|q_t^2 - q_t\| &\leq \|q_t^2 - q_t q\| + \|q_t q - q^2\| + \|q^2 - q\| + \|q - q_t\| \\ &\leq \|q_t - q\|(\|q_t\| + \|q\| + 1) + 4\alpha + \delta \\ &\leq 12\alpha + \delta \\ &< \varepsilon, \end{aligned}$$

and thus q and q' are homotopic in $P_n^{\varepsilon,r}(B_k)$. Therefore, for $p \in P_n^{\varepsilon,r}(A)$ and q in some $M_n(B_k \cap A_r)$ satisfying $\|q - p\| < \frac{\|p^2 - p\|}{12}$, we define $\Upsilon'_{0,\varepsilon,r}([p, l]_{\varepsilon,r})$ to be the image of $[q, l]_{\varepsilon,r}$ in $\lim_k K_*^{\varepsilon,r}(B_k)$. Then $\Upsilon'_{0,\varepsilon,r}$ is a group homomorphism and is an inverse for $\Upsilon_{0,\varepsilon,r}$. We proceed similarly in the odd case. \square

1.5. Morita equivalence. For any unital filtered algebra A , we get an identification between $P_n^{\varepsilon,r}(M_k(A))$ and $P_{nk}^{\varepsilon,r}(A)$ and therefore between $P_\infty^{\varepsilon,r}(M_k(A))$ and $P_\infty^{\varepsilon,r}(A)$. This identification gives rise to a natural group isomorphism between $K_0^{\varepsilon,r}(A)$ and $K_0^{\varepsilon,r}(M_k(A))$, and this isomorphism is induced by the inclusion of C^* -algebras

$$\iota_A : A \hookrightarrow M_k(A); a \mapsto \text{diag}(a, 0).$$

Namely, if we set $e_{1,1} = \text{diag}(1, 0, \dots, 0) \in M_k(\mathbb{C})$, definition of the functoriality yields

$$\iota_{A,*}^{\varepsilon,r}[p, l]_{\varepsilon,r} = [p \otimes e_{1,1} + I_l \otimes (I_k - e_{1,1}), l]_{\varepsilon,r} \in K_0^{\varepsilon,r}(M_k(A))$$

for any p in $P_n^{\varepsilon,r}(A)$ and any integer l with $l \leq n$. We can verify that

$$(\iota_{A,*}^{\varepsilon,r})^{-1}[q, l]_{\varepsilon,r} = [q, kl]_{\varepsilon,r}$$

for any q in $P_n^{\varepsilon,r}(M_k(A))$ and any integer l with $l \leq n$, where on the right hand side of the equality, the matrix q of $M_n(M_k(A))$ is viewed as a matrix of $M_{nk}(A)$.

In a similar way, we obtain in the odd case an identification between $U_\infty^{\varepsilon,r}(M_k(A))$ and $U_\infty^{\varepsilon,r}(A)$ providing a natural semi-group isomorphism between $K_1^{\varepsilon,r}(A)$ and $K_1^{\varepsilon,r}(M_k(A))$. This isomorphism is also induced by the inclusion ι_A and we have

$$\iota_{A,*}^{\varepsilon,r}[x]_{\varepsilon,r} = [x \otimes e_{1,1} + I_n \otimes (I_k - e_{1,1})]_{\varepsilon,r} \in K_1^{\varepsilon,r}(M_k(A))$$

for any x in $U_n^{\varepsilon,r}(A)$.

Let us deal now with the non-unital case. For usual K -theory, Morita equivalence for non-unital C^* -algebra can be deduced from the unital case by using the six-term exact sequence associated to the split extension $0 \rightarrow A \rightarrow \tilde{A} \rightarrow \mathbb{C} \rightarrow 0$. But for quantitative K -theory this splitting only gives rise (in term of section 2.1) to a controlled isomorphism (see corollary 4.9). In order to really have a genuine

isomorphism, we have to go through the tedious following computation. If B is a non-unital C^* -algebra, let us identify $M_k(\tilde{B})$ with $M_k(B) \oplus M_k(\mathbb{C})$ equipped with the product

$$(b, \lambda) \cdot (b', \lambda') = (bb' + \lambda b' + b\lambda', \lambda\lambda')$$

for b and b' in $M_k(B)$ and λ and λ' in $M_k(\mathbb{C})$. Under this identification, if A is not unital, let us check that the semi-group homomorphism

$$\Phi_1 : K_1^{\varepsilon, r}(\tilde{A}) \rightarrow K_1^{\varepsilon, r}(\widetilde{M_k(A)}); [(x, \lambda)]_{\varepsilon, r} \mapsto [(x \otimes e_{1,1}, \lambda)]_{\varepsilon, r}$$

induced by the inclusion ι_A is invertible with inverse given by the composition

$$\Psi_1 : K_1^{\varepsilon, r}(\widetilde{M_k(A)}) \rightarrow K_1^{\varepsilon, r}(M_k(\tilde{A})) \xrightarrow{\cong} K_1^{\varepsilon, r}(\tilde{A}),$$

where the first homomorphism of the composition is induced by the inclusion

$$\widetilde{M_k(A)} \rightarrow M_k(\tilde{A}); (a, z) \mapsto (a, zI_k).$$

Let (x, λ) be an element of $U_n^{\varepsilon, r}(\tilde{A})$, with $x \in M_n(A)$ and $\lambda \in M_n(\mathbb{C})$. Then

$$\Psi_1 \circ \Phi_1[(x, \lambda)]_{\varepsilon, r} = [(x \otimes e_{1,1}, \lambda \otimes I_k)]_{\varepsilon, r},$$

where we use the identification $M_{nk}(\mathbb{C}) \cong M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ to see $x \otimes e_{1,1}$ and $\lambda \otimes I_k$ respectively as matrices in $M_{nk}(A)$ and $M_{nk}(\mathbb{C})$. According to lemma 1.21, as a ε - r -unitary of $M_n(\mathbb{C})$, λ is homotopic to I_n . Hence

$$[(x \otimes e_{1,1}, \lambda \otimes I_k)]_{\varepsilon, r} = [(x \otimes e_{1,1}, \lambda \otimes e_{1,1} + I_n \otimes I_{k-1})]$$

and from this we get that $\Psi_1 \circ \Phi_1$ is induced in K -theory by the inclusion map $\tilde{A} \hookrightarrow M_k(\tilde{A}); a \mapsto \text{diag}(a, 0)$ which is the identity homomorphism (according to the unital case). Conversely, let (y, λ) be an element in $U_n^{\varepsilon, r}(\widetilde{M_k(A)})$ with

$$y \in M_n(M_k(A)) \cong M_n(A) \otimes M_k(\mathbb{C})$$

and $\lambda \in M_n(\mathbb{C})$. Then

$$\Phi_1 \circ \Psi_1[(y, \lambda)]_{\varepsilon, r} = [(y \otimes e_{1,1}, \lambda \otimes I_k)]_{\varepsilon, r},$$

where

- $y \otimes e_{1,1}$ belongs to $M_n(M_k(A)) \otimes M_k(\mathbb{C}) \cong M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$ (the first two factors provide the copy of $M_n(M_k(A))$ where y lies in and $e_{1,1}$ lies in the last factor).
- $\lambda \otimes I_k$ belongs to the algebra $M_n(M_k(\mathbb{C})) \cong M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ that multiplies $M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$ on the first two factors.

Let

$$\sigma : M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C}) \rightarrow M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$$

be the C^* -algebra homomorphism induced by the flip of $M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$. This flip can be realized by conjugation of a unitary U in $M_k(\mathbb{C}) \otimes M_k(\mathbb{C}) \cong M_{k^2}(\mathbb{C})$. Let $(U_t)_{t \in [0,1]}$ be a homotopy in $U_{k^2}(\mathbb{C})$ between U and I_{k^2} . Let us define

$$\mathcal{A} = \{(x, z \otimes I_k); x \in M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C}), z \in M_n(\mathbb{C})\} \subset M_n(\widetilde{M_k(A)}) \otimes M_k(\mathbb{C}),$$

where $z \otimes I_k$ is viewed as $z \otimes I_k \otimes I_k$ in

$$M_n(\widetilde{M_k(A)}) \otimes M_k(\mathbb{C}) \cong M_n(\mathbb{C}) \otimes \widetilde{M_k(A)} \otimes M_k(\mathbb{C}).$$

Then for any $t \in [0, 1]$,

$$\mathcal{A} \rightarrow \mathcal{A}; (x, z \otimes I_k) \mapsto ((I_n \otimes U_t) \cdot x \cdot (I_n \otimes U_t)^{-1}, z \otimes I_k)$$

is an automorphism of C^* -algebra. Hence,

$$((I_n \otimes U_t) \cdot (y \otimes e_{1,1}) \cdot (I_n \otimes U_t^{-1}), \lambda \otimes I_k)_{t \in [0,1]}$$

is a path in $U_{nk}^{\varepsilon,r}(\widetilde{M_k(A)})$ between $(y \otimes e_{1,1}, \lambda \otimes I_k)$ and $(\sigma(y \otimes e_{1,1}), \lambda \otimes I_k)$. The range of $\sigma(y \otimes e_{1,1})$ being in the range of the projection $I_n \otimes e_{1,1} \otimes I_k$, we have an orthogonal sum decomposition

$$(\sigma(y \otimes e_{1,1}), \lambda \otimes I_k) = (\sigma(y \otimes e_{1,1}), \lambda \otimes e_{1,1}) + (0, \lambda \otimes (I_k - e_{1,1}))$$

(recall that $\lambda \otimes e_{1,1}$ and $\lambda \otimes (I_k - e_{1,1})$ multiply $M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$ on the first two factors). By lemma 1.21, λ is homotopic to I_n in $U_n^{\varepsilon,r}(\mathbb{C})$ and thus $(\sigma(y \otimes e_{1,1}), \lambda \otimes I_k)$ is homotopic to $(\sigma(y \otimes e_{1,1}), \lambda \otimes e_{1,1}) + (0, I_n \otimes (I_k - e_{1,1}))$ in $U_{nk}^{\varepsilon,r}(\widetilde{M_k(A)})$ which can be viewed as

$$\text{diag}((y, \lambda), (0, I_{k(k-1)}))$$

in $M_k(M_n(\widetilde{M_k(A)}))$. From this we deduce that $[(y, \lambda)]_{\varepsilon,r} = [(y \otimes e_{1,1}, \lambda \otimes I_k)]_{\varepsilon,r}$ in $K_1^{\varepsilon,r}(\widetilde{M_k(A)})$.

For the even case, by an analogous computation, we can check that the group homomorphisms

$$K_0^{\varepsilon,r}(\tilde{A}) \rightarrow K_0^{\varepsilon,r}(\widetilde{M_k(A)}); [(p, q), l]_{\varepsilon,r} \mapsto [(p \otimes e_{1,1}), q, l]_{\varepsilon,r}$$

and

$$K_0^{\varepsilon,r}(\widetilde{M_k(A)}) \rightarrow K_0^{\varepsilon,r}(\tilde{A}); [(p, q), l]_{\varepsilon,r} \mapsto [(p, q \otimes I_k), kl]_{\varepsilon,r},$$

respectively induce by restriction homomorphisms $\Phi_0 : K_0^{\varepsilon,r}(A) \rightarrow K_0^{\varepsilon,r}(M_k(A))$ and $\Psi_0 : K_0^{\varepsilon,r}(M_k(A)) \rightarrow K_0^{\varepsilon,r}(A)$ which are inverse of each other, where in the right hand side of the last formula, we have viewed $p \in M_n(M_k(A))$ as a matrix in $M_{nk}(A)$ and $q \otimes I_k \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ as a matrix in $M_{nk}(\mathbb{C})$. Since Φ_0 is induced by ι_A , we get from lemma 1.23 that $\iota_{A,*}^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \rightarrow K_0^{\varepsilon,r}(M_k(A))$ is an isomorphism.

Let A be a C^* -algebra filtered by $(A_r)_{r>0}$. Then $\mathcal{K}(\mathcal{H}) \otimes A$ is filtered by $(\mathcal{K}(\mathcal{H}) \otimes A_r)_{r>0}$ and applying proposition 1.28 to the increasing family $(M_k(A)^+)_{k \in \mathbb{N}}$ of C^* -subalgebras of $\widetilde{\mathcal{K}(\mathcal{H}) \otimes A}$, lemmas 1.23 and 1.24, and the discussion above, we deduce the Morita equivalence for $K_*^{\varepsilon,r}(\bullet)$.

Proposition 1.29. *If A is a filtered algebra and \mathcal{H} is a separable Hilbert space, then the homomorphism*

$$A \rightarrow \mathcal{K}(\mathcal{H}) \otimes A; a \mapsto \begin{pmatrix} a & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

induces a $(\mathbb{Z}_2$ -graded) semi-group isomorphism (the Morita equivalence)

$$\mathcal{M}_A^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \rightarrow K_*^{\varepsilon,r}(\mathcal{K}(\mathcal{H}) \otimes A)$$

for any positive number r and any $\varepsilon \in (0, 1/4)$.

1.6. Lipschitz homotopies.

Definition 1.30. *If A is a C^* -algebra and C a positive integer, then a map $h = [0, 1] \rightarrow A$ is called C -Lipschitz if for every t and s in $[0, 1]$, then $\|h(t) - h(s)\| \leq C|t - s|$.*

Proposition 1.31. *There exists a number C such that for any unital filtered C^* -algebra A and any positive number r and $\varepsilon < 1/4$ then :*

- (i) *if p_0 and p_1 are homotopic in $P_n^{\varepsilon, r}(A)$, then there exist integers k and l and a C -Lipschitz homotopy in $P_{n+k+l}^{\varepsilon, r}(A)$ between $\text{diag}(p_0, I_k, 0_l)$ and $\text{diag}(p_1, I_k, 0_l)$.*
- (ii) *if u_0 and u_1 are homotopic in $U_n^{\varepsilon, r}(A)$ then there exist an integer k and a C -Lipschitz homotopy in $U_{n+k}^{3\varepsilon, 2r}(A)$ between $\text{diag}(u_0, I_k)$ and $\text{diag}(u_1, I_k)$.*

Proof.

- (i) Notice first that if p is an ε - r -projection in A , then the homotopy of ε - r -projections of $M_2(A)$ between $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$ in example 1.6 is 2-Lipschitz.

Let $(p_t)_{t \in [0, 1]}$ be a homotopy between p_0 and p_1 in $P_n^{\varepsilon, r}(A)$. Set $\alpha = \inf_{t \in [0, 1]} \frac{\varepsilon - \|p_t^2 - p_t\|}{4}$ and let $t_0 = 0 < t_1 < \dots < t_k = 1$ be a partition of $[0, 1]$ such that $\|p_{t_i} - p_{t_{i-1}}\| < \alpha$ for $i \in \{1, \dots, k\}$. We construct a homotopy of ε - r -projections with the required property between $\text{diag}(p_0, I_{n(k-1)}, 0)$ and $\text{diag}(p_1, I_{n(k-1)}, 0)$ in $M_{n(2k-1)}(A)$ as the composition of the following homotopies.

- We can connect $\text{diag}(p_{t_0}, I_{n(p-1)}, 0)$ and $\text{diag}(p_{t_0}, I_n, 0, \dots, I_n, 0)$ within $P_{n(2k-1)}^{\varepsilon, r}(A)$ by a 2-Lipschitz homotopy.
- As we noticed at the beginning of the proof, we can connect $\text{diag}(p_{t_0}, I_n, 0, \dots, I_n, 0)$ and $\text{diag}(p_{t_0}, I_n - p_{t_1}, p_{t_1}, \dots, I_n - p_{t_k}, p_{t_k})$ within $P_{n(2k-1)}^{\varepsilon, r}(A)$ by a 2-Lipschitz homotopy.
- The ε - r -projections $\text{diag}(p_{t_0}, I_n - p_{t_1}, p_{t_1}, \dots, I_n - p_{t_k}, p_{t_k})$ and $\text{diag}(p_{t_0}, I_n - p_{t_0}, \dots, p_{t_{k-1}}, I_n - p_{t_{k-1}}, p_{t_k})$ satisfy the norm estimate of the assumption of lemma 1.7(i) and hence then can be connected within $P_{n(2k-1)}^{\varepsilon, r}(A)$ by a ray which is clearly a 1-Lipschitz homotopy.
- Using once again the homotopy of example 1.6, we see that $\text{diag}(p_{t_0}, I_n - p_{t_0}, \dots, p_{t_{k-1}}, I_n - p_{t_{k-1}}, p_{t_k})$ and $\text{diag}(0, I_n, \dots, 0, I_n, p_{t_k})$ are connected within $P_{n(2k-1)}^{\varepsilon, r}(A)$ by a 2-Lipschitz homotopy.
- Eventually, $\text{diag}(0, I_n, \dots, 0, I_n, p_{t_k})$ and $\text{diag}(p_{t_k}, I_{n(k-1)}, 0)$ are connected within $P_{n(2k-1)}^{\varepsilon, r}(A)$ by a 2-Lipschitz homotopy.
- (ii) Let $(u_t)_{t \in [0, 1]}$ be a homotopy between u_0 and u_1 in $U_n^{\varepsilon, r}(A)$. Set $\alpha = \inf_{t \in [0, 1]} \frac{\varepsilon - \|u_t^* u_t - I_n\|}{3}$ and let $t_0 = 0 < t_1 < \dots < t_p = 1$ be a partition of $[0, 1]$ such that $\|u_{t_i} - u_{t_{i-1}}\| < \alpha$ for $i \in \{1, \dots, p\}$. We construct a homotopy with the required property between $\text{diag}(u_0, I_{2np})$ and $\text{diag}(u_1, I_{2np})$ within $U_{n(2p+1)}^{3\varepsilon, 2r}(A)$ as the composition of the following homotopies.
 - Since I_{np} and $\text{diag}(u_{t_1}^* u_{t_1}, \dots, u_{t_p}^* u_{t_p})$ satisfy the norm estimate of the assumption of lemma 1.7(ii), then $\text{diag}(u_{t_0}, I_{np})$ is a 3ε - $2r$ -unitary that can be connected to $\text{diag}(u_{t_0}, u_{t_1}^* u_{t_1}, \dots, u_{t_p}^* u_{t_p})$ in $U_{n(p+1)}^{3\varepsilon, 2r}(A)$ by a 1-Lipschitz homotopy.

- Proceeding as in the first point of lemma 1.8, we see that $\text{diag}(I_n, u_{t_1}^*, \dots, u_{t_p}^*, I_{np})$ and $\text{diag}(u_{t_1}^*, \dots, u_{t_p}^*, I_{n(p+1)})$ can be connected within $U_{n(2p+1)}^{\varepsilon, r}(A)$ by a 2-Lipschitz homotopy and thus, in view of remark 1.4,

$$\text{diag}(u_{t_0}, u_{t_1}^* u_{t_1}, \dots, u_{t_p}^* u_{t_p}, I_{np}) =$$

$$\text{diag}(I_n, u_{t_1}^*, \dots, u_{t_p}^*, I_{np}) \cdot \text{diag}(u_{t_0}, u_{t_1}, \dots, u_{t_p}, I_{np})$$

and

$$\text{diag}(u_{t_1}^*, \dots, u_{t_p}^*, I_{n(p+1)}) \cdot \text{diag}(u_{t_0}, u_{t_1}, \dots, u_{t_p}, I_{np}) =$$

$$\text{diag}(u_{t_1}^* u_{t_0}, \dots, u_{t_p}^* u_{t_{p-1}}, u_{t_p}, I_{np})$$

can be connected within $U_{n(2p+1)}^{3\varepsilon, 2r}(A)$ by a 4-Lipschitz homotopy.

- Since $\|u_{t_i}^* u_{t_{i-1}} - I_n\| < \varepsilon$, we get by using once again lemma 1.7(ii) that $\text{diag}(u_{t_1}^* u_{t_0}, \dots, u_{t_p}^* u_{t_{p-1}}, u_{t_p}, I_{np})$ and $\text{diag}(I_{np}, u_{t_p}, I_{np})$ can be connected within $U_{n(2p+1)}^{3\varepsilon, 2r}(A)$ by a 1-Lipschitz homotopy.
- Eventually, $\text{diag}(I_{np}, u_{t_p}, I_{np})$ can be connected to $\text{diag}(u_{t_p}, I_{2np})$ within $U_{(2p+1)n}^{3\varepsilon, 2r}(A)$ by a 2-Lipschitz homotopy.

□

Corollary 1.32. *There exists a control pair (α_h, k_h) such that the following holds:*

For any unital filtered C^ -algebra A , any positive numbers ε and r with $\varepsilon < \frac{1}{4\alpha_h}$ and any homotopic ε - r -projections q_0 and q_1 in $P_n^{\varepsilon, r}(A)$, then there is for some integers k and l an $\alpha_h \varepsilon$ - $k_h \varepsilon$ - r -unitary W in $U_{n+k+l}^{\alpha_h \varepsilon, k_h \varepsilon, r}(A)$ such that*

$$\|\text{diag}(q_0, I_k, 0_l) - W \text{diag}(q_1, I_k, 0_l) W^*\| < \alpha_h \varepsilon.$$

Proof. According to proposition 1.31, we can assume that q_0 and q_1 are connected by a C -Lipschitz homotopy $(q_t)_{t \in [0,1]}$, for some universal constant C . Let $t_0 = 0 < t_1 < \dots < t_p = 1$ be a partition of $[0,1]$ such that $1/32C < |t_i - t_{i-1}| < 1/16C$. With notation of lemma 1.10, pick for every integer i in $\{1, \dots, p\}$ a $\lambda \varepsilon$ - l_ε -unitary W_i in A such that $\|W_i q_{t_{i-1}} W_i^* - q_{t_i}\| < \lambda \varepsilon$. If we set $W = W_p \cdots W_1$, then W is a $3^p \lambda \varepsilon$ - $p l_\varepsilon$ - r -unitary such that $\|W q_0 W^* - q_1\| < 2^p \lambda \varepsilon$. Since $p < 2C$, we get the result. □

2. CONTROLLED MORPHISMS

As we shall see in section 3, usual maps in K -theory such as boundary maps factorize through semi-group homomorphism of quantitative K -theory groups with expansion of norm control and propagation controlled by a control pair. This motivates the notion of controlled morphisms for quantitative K -theory in this section.

Recall that a controlled pair is a pair (λ, h) , where

- $\lambda > 1$;
- $h : (0, \frac{1}{4\lambda}) \rightarrow (0, +\infty)$; $\varepsilon \mapsto h_\varepsilon$ is a map such that there exists a non-increasing map $g : (0, \frac{1}{4\lambda}) \rightarrow (0, +\infty)$, with $h \leq g$.

The set of control pairs is equipped with a partial order: $(\lambda, h) \leq (\lambda', h')$ if $\lambda \leq \lambda'$ and $h_\varepsilon \leq h'_\varepsilon$ for all $\varepsilon \in (0, \frac{1}{4\lambda'})$

2.1. Definition and main properties. For any filtered C^* -algebra A , let us define the families $\mathcal{K}_0(A) = (K_0^{\varepsilon,r}(A))_{0 < \varepsilon < 1/4, r > 0}$, $\mathcal{K}_1(A) = (K_1^{\varepsilon,r}(A))_{0 < \varepsilon < 1/4, r > 0}$ and $\mathcal{K}_*(A) = (K_*^{\varepsilon,r}(A))_{0 < \varepsilon < 1/4, r > 0}$.

Definition 2.1. Let (λ, h) be a controlled pair, let A and B be filtered C^* -algebras, and let i, j be elements of $\{0, 1, *\}$. A (λ, h) -controlled morphism

$$\mathcal{F} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B)$$

is a family $\mathcal{F} = (F^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{4\lambda}, r > 0}$ of semigroups homomorphisms

$$F^{\varepsilon,r} : K_i^{\varepsilon,r}(A) \rightarrow K_j^{\lambda\varepsilon, h\varepsilon r}(B)$$

such that for any positive numbers $\varepsilon, \varepsilon', r$ and r' with $0 < \varepsilon \leq \varepsilon' < \frac{1}{4\lambda}$ and $h\varepsilon r \leq h\varepsilon' r'$, we have

$$F^{\varepsilon',r'} \circ \iota_i^{\varepsilon,\varepsilon',r,r'} = \iota_j^{\lambda\varepsilon, \lambda\varepsilon', h\varepsilon r, h\varepsilon' r'} \circ F^{\varepsilon,r}.$$

If it is not necessary to specify the control pair, we will just say that \mathcal{F} is a controlled morphism.

Let A and B be filtered algebras. Then it is straightforward to check that if $\mathcal{F} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B)$ is a (λ, h) -controlled morphism, then there is group homomorphism $F : K_i(A) \rightarrow K_j(B)$ uniquely defined by $F \circ \iota_i^{\varepsilon,r} = \iota_j^{\lambda\varepsilon, h\varepsilon r} \circ F^{\varepsilon,r}$. The homomorphism F will be called the (λ, h) -controlled homomorphism induced by \mathcal{F} . A homomorphism $F : K_i(A) \rightarrow K_j(B)$ is called (λ, h) -controlled if it is induced by a (λ, h) -controlled morphism. If we don't need to specify the control pair (λ, h) , we will just say that F is a controlled homomorphism.

Example 2.2.

- (i) Let $A = (A_r)_{r>0}$ and $B = (B_r)_{r>0}$ be two filtered C^* -algebras and let $f : A \rightarrow B$ be a homomorphism. Assume that there exists $d > 0$ such that $f(A_r) \subset B_{dr}$ for all positive r . Then f gives rise to a bunch of semigroup homomorphisms $(f_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \rightarrow K_*^{\varepsilon,dr}(B))_{0 < \varepsilon < \frac{1}{4}, r > 0}$ and hence to a $(1, d)$ -controlled morphism $f_* : \mathcal{K}_*(A) \rightarrow \mathcal{K}_*(B)$.
- (ii) The bunch of semi-group isomorphisms

$$(\mathcal{M}_A^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \rightarrow K_*^{\varepsilon,r}(\mathcal{K}(\mathcal{H}) \otimes A))_{0 < \varepsilon < \frac{1}{4}, r > 0}$$

of proposition 1.29 defines a $(1, 1)$ -controlled morphism

$$\mathcal{M}_A : \mathcal{K}_*(A) \rightarrow \mathcal{K}_*(\mathcal{K}(\mathcal{H}) \otimes A)$$

and

$$\mathcal{M}_A^{-1} : \mathcal{K}_*(\mathcal{K}(\mathcal{H}) \otimes A) \rightarrow \mathcal{K}_*(A)$$

inducing the Morita equivalence in K -theory.

If (λ, h) and (λ', h') are two control pairs, define

$$h * h' : (0, \frac{1}{4\lambda\lambda'}) \rightarrow (0, +\infty); \varepsilon \mapsto h_{\lambda'\varepsilon} h'_\varepsilon.$$

Then $(\lambda\lambda', h * h')$ is a control pair. Let A, B_1 and B_2 be filtered C^* -algebras, let i, j and l be in $\{0, 1, *\}$ and let $\mathcal{F} = (F^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{F}}}, r > 0} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B_1)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$ -controlled morphism, let $\mathcal{G} = (G^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{G}}}, r > 0} : \mathcal{K}_j(B_1) \rightarrow \mathcal{K}_l(B_2)$ be a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$ -controlled morphism. Then $\mathcal{G} \circ \mathcal{F} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_l(B_2)$ is the $(\alpha_{\mathcal{G}}\alpha_{\mathcal{F}}, k_{\mathcal{G}} * k_{\mathcal{F}})$ -controlled morphism defined by the family $(\mathcal{G}^{\alpha_{\mathcal{F}}\varepsilon, k_{\mathcal{F}}, \varepsilon r} \circ \mathcal{F}^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{F}}\alpha_{\mathcal{G}}}, r > 0}$.

Remark 2.3. The Morita equivalence for quantitative K -theory is natural, i.e

$$\mathcal{M}_B \circ f = (Id_{\mathcal{K}(\mathcal{H})}) \otimes f \circ \mathcal{M}_A$$

for any homomorphism $f : A \rightarrow B$ of filtered C^* -algebras.

Notation 2.4. Let A and B be filtered C^* -algebras, let (λ, h) be a control pair, and let $\mathcal{F} = (F^{\varepsilon, r})_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{F}}}, r > 0} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B)$ (resp. $\mathcal{G} = (G^{\varepsilon, r})_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{G}}}, r > 0}$) be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$ -controlled morphism (resp. a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$ -controlled morphism). Then we write $\mathcal{F} \stackrel{(\lambda, h)}{\sim} \mathcal{G}$ if

- $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leq (\lambda, h)$ and $(\alpha_{\mathcal{G}}, k_{\mathcal{G}}) \leq (\lambda, h)$.
- for every ε in $(0, \frac{1}{4\lambda})$ and $r > 0$, then

$$\iota_j^{\alpha_{\mathcal{F}}\varepsilon, \lambda\varepsilon, k_{\mathcal{F}}, \varepsilon r, h\varepsilon r} \circ F^{\varepsilon, r} = \iota_j^{\alpha_{\mathcal{G}}\varepsilon, \lambda\varepsilon, k_{\mathcal{G}}, \varepsilon r, h\varepsilon r} \circ G^{\varepsilon, r}.$$

If \mathcal{F} and \mathcal{G} are controlled morphisms such that $\mathcal{F} \stackrel{(\lambda, h)}{\sim} \mathcal{G}$ for a control pair (λ, h) , then \mathcal{F} and \mathcal{G} induce the same morphism in K -theory.

Remark 2.5. Let $\mathcal{F} : \mathcal{K}_i(A_2) \rightarrow \mathcal{K}_j(B_1)$ (resp. $\mathcal{F}' : \mathcal{K}_i(A_2) \rightarrow \mathcal{K}_j(B_1)$) be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$ -controlled (resp. a $(\alpha_{\mathcal{F}'}, k_{\mathcal{F}'})$ -controlled) morphisms and let $\mathcal{G} : \mathcal{K}_{i'}(A_1) \rightarrow \mathcal{K}_i(A_2)$ (resp. $\mathcal{G}' : \mathcal{K}_j(B_1) \rightarrow \mathcal{K}_l(B_2)$) be a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$ -controlled (resp. a $(\alpha_{\mathcal{G}'}, k_{\mathcal{G}'})$ -controlled) morphism. Assume that $\mathcal{F} \stackrel{(\lambda, h)}{\sim} \mathcal{F}'$ for a control pair (λ, h) , then

- $\mathcal{G}' \circ \mathcal{F} \stackrel{(\alpha_{\mathcal{G}'}, \lambda, k_{\mathcal{G}'} * h)}{\sim} \mathcal{G}' \circ \mathcal{F}'$;
- $\mathcal{F} \circ \mathcal{G} \stackrel{(\alpha_{\mathcal{G}}, \lambda, h * k_{\mathcal{G}})}{\sim} \mathcal{F}' \circ \mathcal{G}$.

If i is an element in $\{0, 1, *\}$ and A a filtered C^* -algebra, we denote by $\mathcal{Id}_{\mathcal{K}_i(A)}$ the controlled morphism induced by Id_A .

Let $\mathcal{F} : \mathcal{K}_i(A_1) \rightarrow \mathcal{K}_{i'}(B_1)$, $\mathcal{F}' : \mathcal{K}_j(A_2) \rightarrow \mathcal{K}_l(B_2)$, $\mathcal{G} : \mathcal{K}_i(A_1) \rightarrow \mathcal{K}_j(A_2)$ and $\mathcal{G}' : \mathcal{K}_{i'}(B_1) \rightarrow \mathcal{K}_l(B_2)$ be controlled morphisms and let (λ, h) be a control pair. Then the diagram

$$\begin{array}{ccc} \mathcal{K}_{i'}(B_1) & \xrightarrow{\mathcal{G}'} & \mathcal{K}_l(B_2) \\ \mathcal{F} \uparrow & & \uparrow \mathcal{F}' \\ \mathcal{K}_i(A_1) & \xrightarrow{\mathcal{G}} & \mathcal{K}_j(A_2) \end{array}$$

is called **(λ, h) -commutative** (or **(λ, h) -commutes**) if $\mathcal{G}' \circ \mathcal{F} \stackrel{(\lambda, h)}{\sim} \mathcal{F}' \circ \mathcal{G}$.

Definition 2.6. Let (λ, h) be a control pair, and let $\mathcal{F} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$ -controlled morphism with $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leq (\lambda, h)$.

- \mathcal{F} is called **left (λ, h) -invertible** if there exists a controlled morphism

$$\mathcal{G} : \mathcal{K}_j(B) \rightarrow \mathcal{K}_i(A)$$

such that $\mathcal{G} \circ \mathcal{F} \stackrel{(\lambda, h)}{\sim} \mathcal{Id}_{\mathcal{K}_i(A)}$. The controlled morphism \mathcal{G} is then called a **left (λ, h) -inverse** for \mathcal{F} . Notice that definition of $\stackrel{(\lambda, h)}{\sim}$ implies that $(\alpha_{\mathcal{F}}\alpha_{\mathcal{G}}, k_{\mathcal{F}} * k_{\mathcal{G}}) \leq (\lambda, h)$.

- \mathcal{F} is called **right (λ, h) -invertible** if there exists a controlled morphism

$$\mathcal{G} : \mathcal{K}_j(B) \rightarrow \mathcal{K}_i(A)$$

such that $\mathcal{F} \circ \mathcal{G} \stackrel{(\lambda, h)}{\sim} \mathcal{Id}_{\mathcal{K}_j(B)}$. The controlled morphism \mathcal{G} is then called a **right (λ, h) -inverse** for \mathcal{F} .

- \mathcal{F} is called (λ, h) -invertible or a (λ, h) -isomorphism if there exists a controlled morphism

$$\mathcal{G} : \mathcal{K}_j(B) \rightarrow \mathcal{K}_i(A)$$

which is a left (λ, h) -inverse and a right (λ, h) -inverse for \mathcal{F} . The controlled morphism \mathcal{G} is then called a (λ, h) -inverse for \mathcal{F} (notice that we have in this case necessarily $(\alpha_{\mathcal{G}}, k_{\mathcal{G}}) \leq (\lambda, h)$).

We can check easily that indeed, if \mathcal{F} is left (λ, h) -invertible and right (λ, h) -invertible, then there exists a control pair (λ', h') with $(\lambda, h) \leq (\lambda', h')$, depending only on (λ, h) such that \mathcal{F} is (λ', h') -invertible.

Definition 2.7. Let (λ, h) be a control pair and let $\mathcal{F} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$ -controlled morphism.

- \mathcal{F} is called (λ, h) -injective if $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leq (\lambda, h)$ and for any $0 < \varepsilon < \frac{1}{4\lambda}$, any $r > 0$ and any x in $K_i^{\varepsilon, r}(A)$, then $F^{\varepsilon, r}(x) = 0$ in $K_j^{\alpha_{\mathcal{F}}\varepsilon, k_{\mathcal{F}}, \varepsilon r}(B)$ implies that $\iota_i^{\varepsilon, \lambda\varepsilon, r, h\varepsilon r}(x) = 0$ in $K_i^{\lambda\varepsilon, h\varepsilon r}(A)$;
- \mathcal{F} is called (λ, h) -surjective, if for any $0 < \varepsilon < \frac{1}{4\lambda\alpha_{\mathcal{F}}}$, any $r > 0$ and any y in $K_j^{\varepsilon, r}(B)$, there exists an element x in $K_i^{\lambda\varepsilon, h\varepsilon r}(A)$ such that $F^{\lambda\varepsilon, h\lambda\varepsilon r}(x) = \iota_j^{\varepsilon, \alpha_{\mathcal{F}}\lambda\varepsilon, r, k_{\mathcal{F}}, \lambda\varepsilon h\varepsilon r}(y)$ in $K_j^{\alpha_{\mathcal{F}}\lambda\varepsilon, k_{\mathcal{F}}, \lambda\varepsilon h\varepsilon r}(B)$.

Remark 2.8.

- If $\mathcal{F} : \mathcal{K}_1(A) \rightarrow \mathcal{K}_i(B)$ is a (λ, h) -injective controlled morphism. Then according to lemma 1.16, there exists a control pair (λ', h') with $(\lambda, h) \leq (\lambda', h')$ depending only on (λ, h) such that for any $0 < \varepsilon < \frac{1}{4\lambda'}$, any $r > 0$ and any x and x' in $K_1^{\varepsilon, r}(A)$, then $F^{\varepsilon, r}(x) = F^{\varepsilon, r}(x')$ in $K_i^{\alpha_{\mathcal{F}}\varepsilon, k_{\mathcal{F}}, \varepsilon r}(B)$ implies that $\iota_1^{\varepsilon, \lambda'\varepsilon, r, h'\varepsilon r}(x) = \iota_1^{\varepsilon, \lambda'\varepsilon, r, h'\varepsilon r}(x')$ in $K_1^{\lambda'\varepsilon, h'\varepsilon r}(A)$;
- It is straightforward to check that if \mathcal{F} is left (λ, h) -invertible, then \mathcal{F} is (λ, h) -injective and that if \mathcal{F} is right (λ, h) -invertible, then there exists a control pair (λ', h') with $(\lambda, h) \leq (\lambda', h')$, depending only on (λ, h) such that \mathcal{F} is (λ', h') -surjective.
- On the other hand, if \mathcal{F} is (λ, h) -injective and (λ, h) -surjective, then there exists a control pair (λ', h') with $(\lambda, h) \leq (\lambda', h')$, depending only on (λ, h) such that \mathcal{F} is a (λ', h') -isomorphism.

2.2. Controlled exact sequences.

Definition 2.9. Let (λ, h) be a control pair,

- Let $\mathcal{F} = (F^{\varepsilon, r})_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{F}}}, r > 0} : \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B_1)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$ -controlled morphism, and let $\mathcal{G} = (G^{\varepsilon, r})_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{G}}}, r > 0} : \mathcal{K}_j(B_1) \rightarrow \mathcal{K}_l(B_2)$ be a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$ -controlled morphism, where i, j and l are in $\{0, 1, *\}$ and A, B_1 and B_2 are filtered C^* -algebras. Then the composition

$$\mathcal{K}_i(A) \xrightarrow{\mathcal{F}} \mathcal{K}_j(B_1) \xrightarrow{\mathcal{G}} \mathcal{K}_l(B_2)$$

is said to be (λ, h) -exact at $\mathcal{K}_j(B_1)$ if $\mathcal{G} \circ \mathcal{F} = 0$ and if for any $0 < \varepsilon < \frac{1}{4\max\{\lambda\alpha_{\mathcal{F}}, \alpha_{\mathcal{G}}\}}$, any $r > 0$ and any y in $K_j^{\varepsilon, r}(B_1)$ such that $G^{\varepsilon, r}(y) = 0$ in $K_j^{\alpha_{\mathcal{G}}\varepsilon, k_{\mathcal{G}}, \varepsilon r}(B_2)$, there exists an element x in $K_i^{\lambda\varepsilon, h\varepsilon r}(A)$ such that

$$F^{\lambda\varepsilon, h\lambda\varepsilon r}(x) = \iota_j^{\varepsilon, \alpha_{\mathcal{F}}\lambda\varepsilon, r, k_{\mathcal{F}}, \lambda\varepsilon h\varepsilon r}(y)$$

- in $K_j^{\alpha_{\mathcal{F}}\lambda\varepsilon, k_{\mathcal{F}}, \lambda\varepsilon h_{\varepsilon}r}(B_1)$.
- A sequence of controlled morphisms

$$\cdots \mathcal{K}_{i_{k-1}}(A_{k-1}) \xrightarrow{\mathcal{F}_{k-1}} \mathcal{K}_{i_k}(A_k) \xrightarrow{\mathcal{F}_k} \mathcal{K}_{i_{k+1}}(A_{k+1}) \xrightarrow{\mathcal{F}_{k+1}} \mathcal{K}_{i_{k+2}}(A_{k+2}) \cdots$$
 is called (λ, h) -exact if for every k , the composition

$$\mathcal{K}_{i_{k-1}}(A_{k-1}) \xrightarrow{\mathcal{F}_{k-1}} \mathcal{K}_{i_k}(A_k) \xrightarrow{\mathcal{F}_k} \mathcal{K}_{i_{k+1}}(A_{k+1})$$
 is (λ, h) -exact at $\mathcal{K}_{i_k}(A_k)$.

Remark 2.10. If the composition $\mathcal{K}_i(A) \xrightarrow{\mathcal{F}} \mathcal{K}_1(B_1) \xrightarrow{\mathcal{G}} \mathcal{K}_l(B_2)$ is (λ, h) -exact, then according to lemma 1.16, there exists a control pair (λ', h') with $(\lambda, h) \leq (\lambda', h')$ depending only on (λ, h) , such that for any $0 < \varepsilon < \frac{1}{4 \max\{\lambda' \alpha_{\mathcal{F}}, \alpha_{\mathcal{G}}\}}$, any $r > 0$ and any y and y' in $K_1^{\varepsilon, r}(B_1)$, then $G^{\varepsilon, r}(y) = G^{\varepsilon, r}(y')$ in $K_j^{\alpha_{\mathcal{F}}\varepsilon, k_{\mathcal{F}}, \varepsilon r}(B)$ implies that there exists an element x in $K_i^{\lambda'\varepsilon, h'\varepsilon r}(A)$ such that

$$\iota_j^{\varepsilon, \alpha_{\mathcal{F}}\lambda'\varepsilon, r, k_{\mathcal{F}}, \lambda'\varepsilon h'\varepsilon r}(y') = \iota_j^{\varepsilon, \alpha_{\mathcal{F}}\lambda'\varepsilon, r, k_{\mathcal{F}}, \lambda'\varepsilon h'\varepsilon r}(y) + F^{\lambda'\varepsilon, h'\varepsilon r}(x)$$

in $K_1^{\alpha_{\mathcal{F}}\lambda'\varepsilon, k_{\mathcal{F}}, \lambda'\varepsilon h'\varepsilon r}(B_1)$.

3. EXTENSIONS OF FILTERED C^* -ALGEBRAS

The aim of this section is to establish a controlled exact sequence for quantitative K -theory with respect to extension of filtered C^* -algebras admitting a completely positive cross section that preserves the filtration. We also prove that for these extensions, the boundary maps are induced by controlled morphisms. As in K -theory, one is a map of exponential type and the other is an index type map, and the later in turn fits in a long (λ, h) -controlled exact sequence for some universal control pair (λ, h) .

3.1. Semi-split filtered extensions. Let A be a C^* -algebra filtered by $(A_r)_{r>0}$ and let J be an ideal of A . Then A/J is filtered by $((A/J)_r)_{r>0}$, where $(A/J)_r$ is the image of A_r under the projection $A \rightarrow A/J$. Assume that the C^* -algebra extension

$$0 \rightarrow J \rightarrow A \rightarrow A/J \xrightarrow{q} 0$$

admits a contractive filtered cross-section $s : A/J \rightarrow A$, i.e such that $s((A/J)_r) \subset A_r$ for any positive number. For any $x \in J$ and any number $\varepsilon > 0$ there exists a positive number r and an element a of A_r such that $\|x - a\| < \varepsilon$. Let us set $y = a - s \circ q(a)$. Then y belongs to $A_r \cap J$ and moreover

$$\begin{aligned} \|y - x\| &= \|a - x + s \circ q(x - a)\| \\ &\leq \|a - x\| + \|s \circ q(a - x)\| \\ &\leq \|a - x\| + \|q(x - a)\| \\ &\leq 2\varepsilon. \end{aligned}$$

Hence, $\bigcup_{r>0} (A_r \cap J)$ is dense in J and therefore J is filtered by $(A_r \cap J)_{r>0}$.

Definition 3.1. Let A be a C^* -algebra filtered by $(A_r)_{r>0}$ and let J be an ideal of A . The extension of C^* -algebras

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$$

is said to be filtered and semi-split (or a semi-split extension of filtered C^* -algebras) if there exists a completely positive cross-section

$$s : A/J \rightarrow A$$

such that

$$s((A/J)_r) \subset A_r$$

for any number $r > 0$. Such a cross-section is said to be semi-split and filtered.

We have the following analogous of the lifting property for unitaries of the neutral component.

Lemma 3.2. *There exists a control pair (α_e, k_e) such that for any semi-split extension of filtered C^* -algebras*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{q} A/J \longrightarrow 0$$

with A unital, the following holds: for every positive numbers r and ε with $\varepsilon < \frac{1}{4\alpha_e}$ and any ε - r -unitary V homotopic to I_n in $U_n^{\varepsilon, r}(A/J)$, then for some integer j , there exists a $\alpha_e \varepsilon - k_{e, \varepsilon} r$ -unitary W homotopic to I_{n+j} in $U_{n+j}^{\alpha_e \varepsilon, k_{e, \varepsilon} r}(A)$ and such that $\|q(W) - \text{diag}(V, I_j)\| < \alpha_e \varepsilon$.

Proof. According to proposition 1.31, we can assume that V and I_n are connected by a C -Lipschitz homotopy $(V_t)_{t \in [0, 1]}$, for some universal constant C . Let $t_0 = 0 < t_1 < \dots < t_p = 1$ be a partition of $[0, 1]$ such that $1/16C < |t_i - t_{i-1}| < 1/8C$. Then we get that $\|V_{i-1} - V_i\| < 1/8$ and hence $\|V_{i-1}V_i^* - I_n\| < 1/2$. Let l_ε be the smallest integer such that $\sum_{k \geq l_\varepsilon + 1} 2^{-k}/k < \varepsilon$ and $\sum_{k \geq l_\varepsilon + 1} \log^k 2/k! < \varepsilon$ and let us consider the polynomial functions $P_\varepsilon(x) = \sum_{k=0}^{l_\varepsilon} x^k/k!$ and $Q_\varepsilon(x) = -\sum_{k=1}^{l_\varepsilon} x^k/k$. We get then $\|V_{i-1}V_i^* - P_\varepsilon \circ Q_\varepsilon(1 - V_{i-1}V_i^*)\| \leq 3\varepsilon$. Choose a completely positive section $s(A/J) \rightarrow A$ such that $s(1) = 1$ and let us set $W_i^t = P_\varepsilon(s(tQ_\varepsilon(I_n - V_{i-1}V_i^*)))$ for t in $[0, 1]$ and i in $\{1, \dots, p\}$. Since $V_{i-1}V_i^*$ is closed to the unitary $V_{i-1}V_i^*(V_iV_{i-1}^*V_{i-1}V_i^*)^{-1/2}$, then W_i^t is uniformly (in t and i) closed to $\exp ts(\log(V_{i-1}V_i^*(V_iV_{i-1}^*V_{i-1}V_i^*)^{-1/2}))$ which is unitary (the logarithm is well defined since $V_{i-1}V_i^*(V_iV_{i-1}^*V_{i-1}V_i^*)^{-1/2}$ is closed to I_n) and hence W_i^t is a $\alpha_e 2l_\varepsilon r$ -unitary for some universal α . Hence W_i^1 is a $\alpha_e 2l_\varepsilon r$ -unitary in $U_n^{\varepsilon, r}(A)$ homotopic to I_n and such that $\|q(W_i^1) - V_{i-1}V_i^*\| < 3\alpha_e \varepsilon$. If we set now $W = W_1^1 \dots W_p^1$ and since $p \leq 16C$, then W satisfies the required property. \square

Lemma 3.3. *There exists a control pair (α, k) such that for any semi-split extension of filtered C^* -algebras $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ with A unital, any semi-split filtered cross section $s : A/J \rightarrow A$ with $s(1) = 1$ and any ε - r -projection p in A/J with $0 < \varepsilon < \frac{1}{4\lambda}$, there exists an element y_p in $J_{k_\varepsilon r}$ such that $\|1 + y_p - e^{2i\pi s(k_0(p))}\| < \alpha\varepsilon/3$. In particular $1 + y_p$ is a $\alpha\varepsilon - k_\varepsilon r$ -unitary of J^+ ;*

Proof. Let l_ε be the smallest integer such that

$$\sum_{l=l_\varepsilon+1}^{+\infty} 10^l/l! < \varepsilon.$$

Let us define $z_p = \sum_{l=0}^{l_\varepsilon} \frac{(2i\pi s(p))^l}{l!}$. Then z_p belongs to $M_n(A_{l_\varepsilon r})$ and we have

$$\begin{aligned} \|z_p - e^{2i\pi s(\kappa_0(p))}\| &= \left\| \sum_{l=0}^{l_\varepsilon} \frac{(2i\pi s(p))^l}{l!} - \sum_{l=0}^{+\infty} \frac{(2i\pi s(\kappa_0(p)))^l}{l!} \right\| \\ &\leq \left\| \sum_{l=0}^{l_\varepsilon} \frac{(2i\pi s(p))^l - (2i\pi s(\kappa_0(p)))^l}{l!} \right\| + \left\| \sum_{l=l_\varepsilon+1}^{+\infty} \frac{(2i\pi s(\kappa_0(p)))^l}{l!} \right\| \\ &\leq \|s(p) - s(\kappa_0(p))\| e^{10} + \varepsilon \\ &\leq (2e^{10} + 1)\varepsilon. \end{aligned}$$

If we set $y_p = z_p - s \circ q(z_p)$, then $y_p \in M_n(J \cap A_{l_\varepsilon r})$ and

$$\begin{aligned} \|z_p - (1 + y_p)\| &= \|s \circ q(z_p) - 1\| \\ &\leq \|q(z_p - e^{2i\pi s(\kappa_0(p))})\| \\ &< \lambda\varepsilon, \end{aligned}$$

with $\lambda = (2e^{10} + 1)$. Therefore we have $\|1 + y_p - e^{2i\pi s(\kappa_0(p))}\| < 2\lambda\varepsilon$. The end of the statement is then a consequence of lemma 1.7. \square

3.2. Controlled boundary maps. For any extension $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ of C^* -algebras we denote by $\partial_{J,A} : K_*(A/J) \rightarrow K_*(J)$ the associated (odd degree) boundary map.

Proposition 3.4. *There exists a control pair $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ such that for any semi-split extension of filtered C^* -algebras*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{q} A/J \longrightarrow 0,$$

there exists a $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ -controlled morphism of odd degree

$$\mathcal{D}_{J,A} = (\partial_{J,A}^{\varepsilon,r})_{0 < \varepsilon \leq \frac{1}{4\alpha_{\mathcal{D}}}, r} : K_*(A/J) \rightarrow K_*(J)$$

which induces in K -theory $\partial_{J,A} : K_(A/J) \rightarrow K_*(J)$.*

Proof. Let $s : A/J \rightarrow A$ be a semi-split filtered cross-section. Let us first prove the result when A is unital.

- (i) Let p be an element of $P_n^{\varepsilon,r}(A/J)$. Then $\partial_{J,A}([\kappa_0(p)])$ is the class of $e^{2i\pi s(\kappa_0(p))}$ in $K_1(J)$. Fix a control pair (α, k) as in lemma 3.3 and pick any y_p in $M_n(J_{k_\varepsilon r})$ such that $\|1 + y_p - e^{2i\pi s(\kappa_0(p))}\| < \alpha\varepsilon/3$. Then $1 + y_p$ is an $\alpha\varepsilon - k_\varepsilon r$ -unitary of $M_n(J^+)$, and according to lemma 1.7, any two such $\alpha\varepsilon - k_\varepsilon r$ -unitaries are homotopic in $U_n^{3\alpha\varepsilon, k_\varepsilon r}(J^+)$. Applying lemma 3.3 to $A/J[0, 1]$, we see that the map

$$P_n^{\varepsilon,r}(A/J) \longrightarrow U_n^{3\alpha\varepsilon, k_\varepsilon r}(J^+); p \mapsto 1 + y_p$$

preserves homotopies and hence gives rise to a bunch of well defined semi-group homomorphism

$$\partial_{J,A}^{\varepsilon,r} : K_0^{\varepsilon,r}(A/J) \longrightarrow K_1^{3\alpha\varepsilon, k_\varepsilon r}(J); [p, l]_{\varepsilon,r} \mapsto [1 + y_p]_{3\alpha\varepsilon, k_\varepsilon r}$$

which in the even case satisfies the required properties for a controlled homomorphism.

- (ii) In the odd case, we follow the route of [18, Chapter 8]. For any element u of $U_n^{\varepsilon,r}(A/J)$, pick any element v in some $U_j^{\varepsilon,r}(A/J)$ such that $\text{diag}(u, v)$ is homotopic to I_{n+j} in $U_{n+j}^{3\varepsilon, 2r}(A/J)$ (we can choose in view of lemma 1.16 $v = u^*$). According to lemma 3.2, and up to replace v by $\text{diag}(v, I_k)$ for some integer k , there exists an element w in $U_{n+j}^{3\alpha_e\varepsilon, 2k_e, 3\varepsilon r}(A)$ such that $\|q(w) - \text{diag}(u, v)\| \leq 3\alpha_e\varepsilon$. Let us set $x = w \text{diag}(I_n, 0)w^*$. Then x is an element in $P_{n+j}^{6\alpha_e\varepsilon, 4k_e, 3\varepsilon r}(A)$ such that $\|q(x) - \text{diag}(I_n, 0)\| \leq 9\alpha_e\varepsilon$.

Let us set now $h = x - \text{diag}(I_n, 0) - s \circ q(x - \text{diag}(I_n, 0))$. Then h is a self-adjoint element of $M_{2n}(A_{4k_e, 3\varepsilon r} \cap J)$ such that

$$\|x - \text{diag}(I_n, 0) - h\| \leq 9\alpha_e\varepsilon,$$

and therefore $h + \text{diag}(I_n, 0)$ belongs to $P_{n+j}^{45\alpha_e\varepsilon, 4k_e, 3\varepsilon r}(J)$. Define then

$$\partial_{J,A}^{\varepsilon,r}([u]_{\varepsilon,r}) = [h + \text{diag}(I_n, 0), n]_{450\alpha_e\varepsilon, 4k_e, 3\varepsilon r}.$$

It is straightforward to check that (compare with [18, Chapter 8]).

- two choice of elements satisfying the conclusion of lemma 3.2 relatively to $\text{diag}(u, v)$ give rise to homotopic elements $P_{n+j}^{450\alpha_e\varepsilon, 4k_e, 3\varepsilon r}(J)$ (this is a consequence of lemma 1.7).
- Replacing u by $\text{diag}(u, I_m)$ and v by $\text{diag}(v, I_k)$ gives also rise to the same element of $K_0^{450\alpha_e\varepsilon, 4k_e, 3\varepsilon r}(J)$.

Applying now lemma 3.2 to the exact sequence

$$0 \rightarrow J[0, 1] \rightarrow A[0, 1] \rightarrow A/J[0, 1] \rightarrow 0,$$

we get that $\partial_{J,A}^{\varepsilon,r}([u]_{\varepsilon,r})$

- only depends on the class of u in $K_1^{\varepsilon,r}(A/J)$;
- does not depend on the choice of v such that $\text{diag}(u, v)$ is connected to I_{n+j} in $U_{n+j}^{\varepsilon,r}(A/J)$.
- If A is not unital, use the exact sequence

$$0 \rightarrow J \rightarrow \tilde{A} \rightarrow \widetilde{A/J} \rightarrow 0$$

to define $\partial_{J,A}^{\varepsilon,r}$ as the composition

$$K_0^{\varepsilon,r}(A/J) \hookrightarrow K_0^{\varepsilon,r}(\widetilde{A/J}) \xrightarrow{\partial_{J,\tilde{A}}^{\varepsilon,r}} K_1^{450\alpha_e\varepsilon, 4k_s, 3\varepsilon r}(J),$$

where the inclusion in the composition is induced by the inclusion $A/J \hookrightarrow \widetilde{A/J} \cong \tilde{A}/J$.

- Since the set of filtered semi-split cross-section $s : A/J \rightarrow A$ such that $s((A/J)_r) \subset A_r$ is convex, the definition of $\partial_{J,A}^{\varepsilon,r}$ actually does not depend on the choice of such a section.
- Using lemma 1.7, it is plain to check that for a suitable control pair $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$, then $\mathcal{D}_{J,A} = (\partial_{J,A}^{\varepsilon,r})_{0 < \varepsilon \leq \frac{1}{4\alpha_{\mathcal{D}}}, r}$ is a $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ -controlled morphism inducing the (odd degree) boundary map $\partial_{J,A} : K_*(A/J) \rightarrow K_*(J)$.

□

For a semi-split extension of filtered C^* -algebras

$$0 \longrightarrow J \longrightarrow A \xrightarrow{q} A/J \longrightarrow 0,$$

we set $\mathcal{D}_{J,A}^0 : \mathcal{K}_0(A/J) \rightarrow \mathcal{K}_1(J)$, for the restriction of $\mathcal{D}_{J,A}$ to $\mathcal{K}_0(A/J)$ and $\mathcal{D}_{J,A}^1 : \mathcal{K}_1(A/J) \rightarrow \mathcal{K}_0(J)$, for the restriction of $\mathcal{D}_{J,A}$ to $\mathcal{K}_1(A/J)$.

Remark 3.5.

- (i) *Let A and B be two filtered C^* -algebras and let $\phi : A \rightarrow B$ be a filtered homomorphism. Let I and J be respectively ideals in A and B and assume that*
 - $\phi(I) \subset J$;
 - *there exists semi-split filtered cross-sections $s : A/I \rightarrow A$ and $s' : B/J \rightarrow J$ such that $s' \circ \tilde{\phi} = \phi \circ s$, where $\tilde{\phi} : A/I \rightarrow B/J$ is the homomorphism induced by ϕ ,**then $\mathcal{D}_{J,B} \circ \tilde{\phi}_* = \phi_* \circ \mathcal{D}_{I,A}$.*
- (ii) *Let $0 \rightarrow J \rightarrow A \xrightarrow{q} A/J \rightarrow 0$ be a split extension of filtered C^* -algebras, i.e there exists a homomorphism of filtered C^* -algebras $s : A/J \rightarrow A$ such that $q \circ s = \text{Id}_{A/J}$. Then we have $\mathcal{D}_{J,A} = 0$.*

For a filtered C^* -algebra A , we have defined the suspension and the cone respectively as $SA = C_0((0, 1), A)$ and $CA = C_0([0, 1], A)$. Then SA and CA are filtered C^* -algebras and evaluation at the value 1 gives rise to a semi-split filtered extension of C^* -algebras

$$(1) \quad 0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0$$

and in the even case, the corresponding boundary $\partial_{SA,CA} : K_0(A) \rightarrow K_1(SA)$ implements the suspension isomorphism and has the following easy description when A is unital: if p is a projection, then $\partial_{SA,CA}[p]$ is the class in $K_1(SA)$ of the path of unitaries

$$[0, 1] \rightarrow U_n(A); t \mapsto pe^{2i\pi t} + 1 - p.$$

Let us show that we have an analogous description in term of almost projection. Notice that if q is an ε - r -projection in A , then

$$z_q : [0, 1] \rightarrow A; t \mapsto qe^{2i\pi t} + 1 - q$$

is a 5ε - r -unitary in \widetilde{SA} . Using this, we can define a $(5, 1)$ -controlled morphism $\mathcal{Z}_A = (Z_A^{\varepsilon,r})_{0 < \varepsilon < 1/20, r > 0} : \mathcal{K}_0(A) \rightarrow \mathcal{K}_1(SA)$ in the following way:

- for any q in $P_n^{\varepsilon,r}(A)$ and any integer k let us set
- $$V_{q,k} : [0, 1] \rightarrow U_n^{\varepsilon,r}(\widetilde{SA}) : t \mapsto \text{diag}(e^{-2ki\pi t}, 1, \dots, 1) \cdot (1 - q + qe^{2i\pi t});$$
- define then $Z_A^{\varepsilon,r}([q, k]_{\varepsilon,r}) = [V_{q,k}]_{5\varepsilon,r}$.

Proposition 3.6. *There exists a control pair (λ, h) such that for any filtered C^* -algebra A , then $\mathcal{D}_{CA,SA}^0 \stackrel{(\lambda,h)}{\sim} \mathcal{Z}_A$.*

Proof. Let $[q, k]_{\varepsilon,r}$ be an element of $K_0^{\varepsilon,r}(A)$, with q in $P_n^{\varepsilon,r}(A)$ and k integer. We can assume without loss of generality that $n \geq k$. Namely, up to replace n by $2n$ and using a homotopy between $\text{diag}(q, 0)$ and $\text{diag}(0, q)$ in $P_{2n}^{\varepsilon,r}(A)$, we can indeed assume that q and $\text{diag}(I_k, 0)$ commute. As in the proof of proposition 3.4, define l_ε as the smallest integer such that $\sum_{l=l_\varepsilon+1}^\infty 10^l/l! < \varepsilon$. Let us consider the following paths in $M_n(A)$

$$z : [0, 1] \rightarrow M_n(A); t \mapsto \sum_{l=0}^{l_\varepsilon} (2i\pi(tq + (1-t)\text{diag}(I_k, 0)))^l / l!$$

and

$$z' : [0, 1] \longrightarrow M_n(A); t \mapsto \exp(2i\pi \operatorname{diag}(-tI_k, 0))(1 - q + e^{2i\pi t}q).$$

Since q and I_k commutes, then

$$\exp(2i\pi(\operatorname{diag}(-tI_k, 0) + tq)) = \exp(2i\pi \operatorname{diag}(-tI_k, 0)) \cdot \exp(2i\pi tq)$$

and hence

$$z(t) = \exp(2i\pi \operatorname{diag}(-tI_k, 0)) \exp(2i\pi tq) - \sum_{l=l_\varepsilon+1}^{\infty} (2i\pi(tq + (1-t) \operatorname{diag}(I_k, 0)))^l / l!.$$

We get therefore

$$\begin{aligned} \|z(t) - z'(t)\| &\leq \varepsilon + \|qe^{2i\pi t} + (1-q) - \exp 2i\pi tq\| \\ &\leq \varepsilon + 2\|\kappa_0(q) - q\| + \|\exp 2i\pi t\kappa_0(q) - \exp 2i\pi tq\| \\ &\leq \varepsilon(5 + 4e^{4\pi}). \end{aligned}$$

Let us set

$$y : [0, 1] : \longrightarrow M_n(A); t \mapsto z(t) - 1 - (1-t) \operatorname{diag}(I_k, 0) \sum_{l=1}^{l_\varepsilon} (2i\pi)^l / l! - t \sum_{l=1}^{l_\varepsilon} (2i\pi q)^l / l!.$$

For some $\alpha_s \geq \alpha_\partial$, we get then that $1 + y$ and z' are homotopic elements in $U_n^{\alpha_s \varepsilon, k_{\partial, \varepsilon} r}(\widetilde{SA})$. Using the semi-split filtered cross-section $A \rightarrow CA; a \mapsto [t \mapsto ta]$ for the extension of equation (1), we get in view of the proof of proposition 3.4,

$$l_1^{\alpha_\partial \varepsilon, \alpha_s \varepsilon, k_{\partial, \varepsilon} r} \circ \partial_{SA, CA}^{\varepsilon, r}([q, k]_{\varepsilon, r}) = [1 + y]_{\alpha_s \varepsilon, k_{\partial, \varepsilon} r},$$

and thus we deduce

$$l_1^{\alpha_\partial \varepsilon, \alpha_s \varepsilon, k_{\partial, \varepsilon} r} \circ \partial_{SA, CA}^{\varepsilon, r}([q, k]_{\varepsilon, r}) = [z']_{\alpha_s \varepsilon, k_{\partial, \varepsilon} r}.$$

We get the result by using a homotopy of unitaries in $M_n(\widetilde{SA})$ between

$$t \mapsto \operatorname{diag}(e^{-2k\pi t}, 1, \dots, 1)$$

and $t \mapsto \exp(2i\pi \operatorname{diag}(-tI_k, I_{n-k}))$. \square

The inverse of the suspension isomorphism is provided, up to Morita equivalence by the Toeplitz extension: let us consider the unilateral shift S on $\ell^2(\mathbb{N})$, i.e the operator defined on the canonical basis $(e_n)_{n \in \mathbb{N}}$ of $\ell^2(\mathbb{N})$ by $S(e_n) = e_{n+1}$ for all integer n . Then the Toeplitz algebra \mathcal{T} is the C^* -subalgebra of $\mathcal{L}(\ell^2(\mathbb{N}))$ generated by S . The algebra of compact operators $\mathcal{K}(\ell^2(\mathbb{N}))$ is an ideal of \mathcal{T} and we get an extension of C^* -algebras

$$0 \rightarrow \mathcal{K}(\ell^2(\mathbb{N})) \rightarrow \mathcal{T} \xrightarrow{\rho} C(\mathbb{S}_1) \rightarrow 0,$$

called the Toeplitz extension, where \mathbb{S}_1 denote the unit circle. Let us define $\mathcal{T}_0 = \rho^{-1}(C_0((0, 1)))$, where $C_0((0, 1))$ is viewed as a subalgebra of $C(\mathbb{S}_1)$. We obtain then an extension of C^* -algebras

$$0 \rightarrow \mathcal{K}(\ell^2(\mathbb{N})) \rightarrow \mathcal{T}_0 \xrightarrow{\rho} C_0(0, 1) \rightarrow 0.$$

For any C^* -algebra A , we can tensorize this exact sequence to obtain an extension

$$0 \rightarrow \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \rightarrow \mathcal{T}_0 \otimes A \rightarrow SA \rightarrow 0$$

which is filtered and semi-split when A is a filtered C^* -algebra.

Proposition 3.7. *There exists a control pair (λ, h) such that*

$$\mathcal{D}_{\mathcal{K}(\ell^2(\mathbb{N})) \otimes A, \mathcal{T}_0 \otimes A}^1 \circ \mathcal{Z}_A \stackrel{(\lambda, h)}{\sim} \mathcal{M}_A$$

for any unital filtered C^* -algebra A .

Proof. Let q be a ε - r -projection in $M_n(A)$. We can assume indeed without loss of generality that $n = 1$. The Toeplitz extension is semi-split by the section induced by the completely positive map $s : C(\mathbb{S}_1) \rightarrow \mathcal{T}; f \mapsto M_f$, where if π_0 stands for the projection $L^2(\mathbb{S}_1) \cong \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{N})$, then M_f is the composition

$$\ell^2(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{Z}) \cong L^2(\mathbb{S}_1) \xrightarrow{f} L^2(\mathbb{S}_1) \xrightarrow{\pi_0} \ell^2(\mathbb{N}),$$

($f \cdot$ being the pointwise multiplication by f). Notice first that $\begin{pmatrix} S & 1 - SS^* \\ 0 & S^* \end{pmatrix}$ is a unitary lift of $\mathbb{S}_1 \rightarrow M_2(\mathbb{C}); z \mapsto \text{diag}(z, \bar{z})$ in $M_2(\mathcal{T})$ under the homomorphism induced by $\rho : \mathcal{T} \rightarrow C(\mathbb{S}_1)$. Under the section induced by s , we see that z_q lifts to $1 \otimes (1 - q) + S \otimes q$, and hence

$$W = \begin{pmatrix} S & 1 - SS^* \\ 0 & S^* \end{pmatrix} \otimes q + I_2 \otimes (1 - q)$$

is a lift in $U_2^{5\varepsilon, r}(\mathcal{T}_0 \otimes A)$ of $\text{diag}(z_q, z_q^*)$. Since $\|q(1 - q)\| < \varepsilon$, we see that $W^* \text{diag}(1, 0)W$ is closed to

$$\begin{pmatrix} S^* & 0 \\ 1 - SS^* & S \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S & 1 - SS^* \\ 0 & S^* \end{pmatrix} \otimes q^2 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (1 - q)^2.$$

Hence, $W^* \text{diag}(1, 0)W$ is an element of $P_2^{10\varepsilon, 2r}(\mathcal{T}_0 \otimes A)$ which is closed to $\text{diag}(1, (1 - SS^*) \otimes q)$. Since

$$\mathcal{M}_A([q, 0]_{\varepsilon, r}) = [\text{diag}(0, (1 - SS^*) \otimes q)]_{\varepsilon, r},$$

we get the existence of a positive real α_t such that the proposition holds. \square

3.3. Long exact sequence. We follow the route of [18, Sections 6.3, 7.1 and 8.2] to state for semi-split extensions of filtered C^* -algebras (λ, h) -exact long exact sequences in quantitative K -theory, for some universal control pair (λ, h) .

Proposition 3.8. *There exists a control pair (λ, h) such that for any semi-split extension of filtered C^* -algebras*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} A/J \longrightarrow 0,$$

the composition

$$\mathcal{K}_*(J) \xrightarrow{j_*} \mathcal{K}_*(A) \xrightarrow{q_*} \mathcal{K}_*(A/J)$$

is (λ, h) -exact at $\mathcal{K}_*(A)$.

Proof. We can assume without loss of generality that A is unital. In the even case, let y be an element of $K_0^{\varepsilon, r}(A)$ such that $q_*(y) = 0$ in $K_0^{\varepsilon, r}(A/J)$, let e be an ε - r -projection in $M_n(A)$ and let l be a positive integer such that $y = [e, k]_{\varepsilon, r}$. Up to stabilization, we can assume that $k \leq n$ and that $q(e)$ is homotopic to $p_k = \text{diag}(I_k, 0)$ as an ε - r -projection in $M_n(A/J)$. According to corollary 1.32, there exists up to stabilization a $\alpha_h \varepsilon$ - $k_{h, \varepsilon} r$ -unitary W of $M_n(A/J)$ such that

$$\|Wq(e)W^* - p_k\| \leq \alpha_h \varepsilon.$$

The $3\alpha_s\varepsilon-2k_{h,\varepsilon}r$ -unitary $\text{diag}(W, W^*)$ of $M_{2n}(A/J)$ is homotopic to I_{2n} . Let choose as in lemma 3.2, a control pair (α, l) , an integer j and a $\alpha\varepsilon-l_\varepsilon r$ -unitary V of $M_{2n+j}(A)$ such that

$$\|q(V) - \text{diag}(W, W^*, I_{k+j})\| \leq \alpha\varepsilon.$$

If we set $e' = V \text{diag}(e, 0) V^*$, then e' is a $4\alpha\varepsilon-2l_\varepsilon r$ -projection in $M_{2n+j}(A)$. If $s : A/J \rightarrow A$ is a semi-split filtered cross-section such that $s(1) = 1$, define $f = e' - s \circ q(e' - \text{diag}(I_n, 0))$. We see that f belongs to $M_{2n+j}(J^+)$ and moreover, since $\|f - e'\| \leq (4\alpha + \alpha_h)\varepsilon$, then according to lemma 1.7, f is for a suitable λ a $\lambda\varepsilon-2l_\varepsilon r$ -projection of $M_{2n+k}(J^+)$ homotopic to e' . Then $x = [f, k]_{\lambda\varepsilon, 2l_\varepsilon r}$ defines a class in $K_0^{\lambda\varepsilon, 2l_\varepsilon r}(J)$. As in the proof of (ii) of lemma 1.8 we can choose λ big enough so that $\text{diag}(e', I_{2n+j})$ and $\text{diag}(e, 0, I_{2n+j})$ are homotopic $\lambda\varepsilon-2k_{h,\varepsilon}r$ -projections of $M_{2n}(A)$ and hence we get the result in the even case.

For the odd case, let y be an element in $K_1^{\varepsilon, r}(A)$ such that $q_*(y) = 0$ in $K_1^{\varepsilon, r}(A/J)$ and let us choose an ε - r -unitary V in some $M_n(A)$ such that $y = [V]_{\varepsilon, r}$. In view of lemma 3.2 and up to enlarge the size of the matrix V , we can assume that $\|q(V) - q(W)\| \leq \alpha_e\varepsilon$ with W a $\alpha_e\varepsilon-k_{e,\varepsilon}r$ -unitary of $M_n(A)$ homotopic to I_n . Hence W^*V and V are homotopic $3\alpha_e\varepsilon-(k_{e,\varepsilon}+1)r$ -unitary of $M_n(A)$. If we set

$$U = W^*V + s \circ q(I_n - W^*V),$$

then the coefficients of the matrix $U - I_n$ lie in J . Moreover, since

$$\|U - W^*V\| \leq (2\alpha_e + 1)\varepsilon,$$

we obtain that U is a $\lambda\varepsilon-(l_\varepsilon+1)r$ -unitary for some $\lambda \geq 1$. Hence, $x = [U]_{\lambda\varepsilon, (k_{e,\varepsilon}+1)r}$ defines a class in $K_1^{\lambda\varepsilon, (k_{e,\varepsilon}+1)r}(J)$ with the required property. \square

Proposition 3.9. *There exists a control pair (λ, h) such that for any semi-split extension of filtered C^* -algebras*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} A/J \longrightarrow 0,$$

the composition

$$\mathcal{K}_1(A) \xrightarrow{q_*} \mathcal{K}_1(A/J) \xrightarrow{\mathcal{D}_{J,A}^1} \mathcal{K}_0(J)$$

is (λ, h) -exact at $\mathcal{K}_1(A/J)$.

Proof. We can assume without loss of generality that A is unital. Let y be an element of $K_1^{\varepsilon, r}(A/J)$ such that $\partial_{J,A}^{\varepsilon, r}(y) = 0$ in $K_0^{\alpha\varepsilon, k_{\partial, \varepsilon}r}(A/J)$ and let U be an ε - r -unitary of $M_n(A/J)$ such that $y = [U]_{\varepsilon, r}$. With notation of lemma 3.2, let j be an integer and W be a $3\alpha_e\varepsilon-2k_{e,3\varepsilon}r$ -unitary in $M_{2n+j}(A)$ such that

$$\|q(W) - \text{diag}(U, U^*, I_j)\| \leq \alpha\varepsilon.$$

Set $x = W \text{diag}(I_n, 0) W^*$ and $h = x - \text{diag}(I_n, 0) - s \circ q(x - \text{diag}(I_n, 0))$ as in the proof of proposition 3.4. Since $\partial_{J,A}^{\varepsilon, r}(y) = 0$, we can up to take a larger n assume that $h + \text{diag}(I_n, 0)$ is homotopic to $\text{diag}(I_n, 0)$ as an $\alpha_{\mathcal{D}\varepsilon-k_{\mathcal{D}, \varepsilon}r}$ -projection of $M_{2n+j}(\tilde{J})$. Since x is close to $h + \text{diag}(I_n, 0)$, we get from corollary 1.32 that up to take a larger j , there exists for a control pair (α, l) , depending only on the control pairs (α_h, k_h) and $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ of corollary 1.32 and lemma 3.3, an $\alpha\varepsilon-l_\varepsilon r$ -unitary V' in $M_{2n+j}(\tilde{J})$ such that

$$\|W \text{diag}(I_n, 0) W^* - V' \text{diag}(I_n, 0) V'^*\| \leq \alpha\varepsilon.$$

Then $V = \rho_J(V')V'^{-1}W^*$ is a $10(\alpha + \alpha_\varepsilon)\varepsilon\text{--}(l_\varepsilon + k_{e,\varepsilon})r$ -unitary in $M_{2n+j}(A)$ such that

$$\|q(V) - \text{diag}(U, U^*, I_j)\| \leq \alpha\varepsilon.$$

Since for a suitable constant α' depending only on α we have

$$\|\rho_J(V') \text{diag}(I_n, 0) \rho_J(V'^*) - \text{diag}(I_n, 0)\| \leq \alpha'\varepsilon,$$

we obtain that

$$\|V \text{diag}(I_n, 0) V^* - \text{diag}(I_n, 0)\| \leq \alpha''\varepsilon$$

and

$$\|V^* \text{diag}(I_n, 0) V - \text{diag}(I_n, 0)\| \leq \alpha''\varepsilon$$

for some constant α'' depending only on α' that we can choose indeed larger than $(10\alpha + \alpha_\varepsilon)$. Hence the $n \times n$ -left upper corner X of V is a $\alpha''\varepsilon\text{--}(l_\varepsilon + l'_\varepsilon)r$ -unitary in $M_n(A)$ such that $\|q(X) - U\| \leq \alpha''\varepsilon$. Hence we get the result. \square

Proposition 3.10. *There exists a control pair (λ, h) such that for any semi-split extension of filtered C^* -algebras*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} A/J \longrightarrow 0,$$

the composition

$$\mathcal{K}_1(A/J) \xrightarrow{\mathcal{D}_{J,A}^1} \mathcal{K}_0(J) \xrightarrow{j_*} \mathcal{K}_0(A)$$

is (λ, h) -exact in $\mathcal{K}_0(J)$.

Proof. It is enough to prove the result for A unital. Let y be an element of $K_0^{\varepsilon,r}(J)$ such that $j_*^{\varepsilon,r}(y) = 0$ in $K_0^{\varepsilon,r}(A)$, let e be an ε - r -projection in $M_n(J^+)$ and k be a positive integer such that $y = [e, k]_{\varepsilon,r}$. If we set $p_k = \text{diag}(I_k, 0)$, we can indeed assume without loss of generality that $\|q(e) - p_k\| \leq 2\varepsilon$ (where J^+ is viewed as a subalgebra of A). Up to stabilization, we can also assume that e is homotopic to p_k as an ε - r -projection in $M_n(A)$. According to corollary 1.32, there exists up to stabilization a $\alpha_h\varepsilon\text{--}k_{h,\varepsilon}r$ -unitary W of $M_n(A)$ such that

$$\|e - Wp_kW^*\| \leq \alpha_h\varepsilon.$$

Up to replace n by $2n$, W by $\text{diag}(W, W^*)$ and e by $\text{diag}(e, 0)$, we can assume that W is a $3\alpha_h\varepsilon\text{--}2k_{h,\varepsilon}r$ -unitary homotopic to I_n . Since

$$\begin{aligned} \|q(W)p_kq(W^*) - p_k\| &\leq \|q(W)p_kq(W^*) - q(e)\| + \|q(e) - p_k\| \\ &< (2 + \alpha_h)\varepsilon, \end{aligned}$$

then

$$\|q(W^*)p_kq(W) - p_k\| < (2 + 4\alpha_h)\varepsilon.$$

Hence for an $\alpha' > 1$ depending only on α_h , the left-up $n \times n$ corner V_1 and the right bottom corner V_2 of $q(W)$ are $\alpha'\varepsilon\text{--}2k_{e,\varepsilon}r$ -unitaries of $M_n(A/J)$ such that

$$\|q(W)q(W^*) - \text{diag}(V_1, V_2) \text{diag}(V_1, V_2)^*\| < (\alpha_h + \alpha')\varepsilon$$

and

$$\|q(W^*)q(W) - \text{diag}(V_1, V_2)^* \text{diag}(V_1, V_2)\| < (\alpha_h + \alpha')\varepsilon.$$

Hence $q(W)$ is close to $\text{diag}(V_1, V_2)$ and hence there is a $\lambda > 1$ depending only on α_e such that as a $\lambda\varepsilon\text{--}2k_{h,\varepsilon}r$ -unitary of $M_n(A/J)$, then $\text{diag}(V_1, V_2)$ is homotopic

to $q(W)$ and hence to I_n . We can indeed choose λ big enough such that if we set $x = [V_1]_{\lambda\varepsilon, 2k_{e,\varepsilon}r}$, then

$$\begin{aligned}\partial_{J,A}^{\lambda\varepsilon, 2k_{e,\varepsilon}r}(x) &= [e, k]_{\lambda\alpha_\beta\varepsilon, k_{\partial,\alpha\varepsilon}2k_{e,\varepsilon}r} \\ &= \iota_*^{\varepsilon, r, \lambda\varepsilon, 2k_{e,\varepsilon}r}(y).\end{aligned}$$

□

From propositions 3.8, 3.9 and 3.10 we can derive the analogue of the long exact sequence in K -theory.

Theorem 3.11. *There exists a control pair (λ, h) such that for any semi-split extension of filtered C^* -algebras*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} A/J \longrightarrow 0,$$

the sequence

$$\mathcal{K}_1(J) \xrightarrow{j_*} \mathcal{K}_1(A) \xrightarrow{q_*} \mathcal{K}_1(A/J) \xrightarrow{\mathcal{D}_{J,A}^1} \mathcal{K}_0(J) \xrightarrow{j_*} \mathcal{K}_0(A) \xrightarrow{q_*} \mathcal{K}_0(A/J)$$

is (λ, h) -exact.

As a consequence, using the exact sequence

$$(2) \quad 0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0,$$

and in view of lemma 1.27 and point (iii) of remark 2.8, we deduce in the setting of the semigroup $K_*^{\varepsilon, r}(\bullet)$ the analogue of the suspension isomorphism in K -theory.

Corollary 3.12. *Let $\mathcal{D}_A^1 = \mathcal{D}_{SA, CA}^1 : \mathcal{K}_1(A) \rightarrow \mathcal{K}_0(SA)$ be the controlled boundary morphism associated to the semi-split and filtered extension of equation (2) for a filtered C^* -algebra A .*

- *There exists a control pair (λ, h) such that for any filtered C^* -algebra A , then \mathcal{D}_A^1 is (λ, h) -invertible.*
- *Moreover, we can choose a (λ, h) -inverse which is natural: there exists a control pair (α_β, k_β) and for any filtered C^* -algebra A a (λ, h) -controlled morphism $\mathcal{B}_A^0 = (\beta_A^{\varepsilon, r})_{0 < \varepsilon < \frac{1}{4\alpha_\beta}, r > 0} : \mathcal{K}_0(SA) \rightarrow \mathcal{K}_1(A)$ which is an (λ, h) -inverse for \mathcal{D}_A^1 and such that $\mathcal{B}_B^0 \circ f_S = f \circ \mathcal{B}_A^0$ for any homomorphism $f : A \rightarrow B$ of filtered C^* -algebras, where $f_S : SA \rightarrow SB$ is the suspension of the homomorphism f .*

3.4. The mapping cones. We end this section by proving that the mapping cones construction can be performed in the framework of quantitative K -theory. Let

$$0 \rightarrow J \rightarrow A \xrightarrow{q} A/J \rightarrow 0$$

be a filtered semi-split extension of C^* -algebras. Let us set $A/J[0, 1) = C_0([0, 1), A/J)$ and define the mapping cone of q :

$$C_q = \{(x, f) \in A \oplus A/J[0, 1); \text{ such that } f(0) = q(x)\}.$$

Using a semi-split filtered cross-section for q , we see that C_q is filtered by

$$(C_q \cap (A_r \oplus A/J[0, 1)))_{r > 0}.$$

Let us set

$$e_q : J \rightarrow C_q; x \mapsto (x, 0)$$

and

$$\phi_q : SA/J \rightarrow C_q; f \mapsto (0, f).$$

We have then a semi-split extension of filtered C^* -algebras

$$0 \rightarrow J \xrightarrow{e_j} C_q \xrightarrow{\pi_2} A/J[0, 1] \rightarrow 0,$$

where π_2 is the projection on the second factor of $A \oplus A/J[0, 1]$.

Lemma 3.13. *There exists a control pair (λ, h) such that $e_{q,*}$ is (λ, h) -invertible for any semi-split extension of filtered C^* -algebras $0 \rightarrow J \rightarrow A \xrightarrow{q} A/J \rightarrow 0$.*

Proof. The even case is a consequence of theorem 3.11. We deduce the odd case from the even one using corollary 3.12. \square

It is a standard fact in K -theory that the boundary of an extension of C^* -algebras $0 \rightarrow J \rightarrow A \xrightarrow{q} A/J \rightarrow 0$ can be obtained using the equality

$$e_{q,*} \circ \partial_{J,A} = \phi_{q,*} \circ \partial_{A/J},$$

where $\partial_{A/J} = \partial_{SA/J, CA/J}$ stands for the boundary map of the extension

$$0 \rightarrow SA/J \rightarrow CA/J \rightarrow A/J \rightarrow 0$$

(corresponding to the evaluation at 1). We have a similar result in quantitative K -theory:

Lemma 3.14. *With above notations, we have $e_{q,*} \circ \mathcal{D}_{J,A} = \phi_{q,*} \circ \mathcal{D}_{A/J}$, where $\mathcal{D}_{A/J}$ stands for $\mathcal{D}_{SA/J, CA/J}$.*

Proof. We can assume without loss of generality that A is unital. Let us fix a semi-split filtered cross-section $s : A/J \rightarrow A$ such that $s(1) = 1$. Let p be an ε - r projection in A/J . Using the notations of the proof of proposition 3.3, define for t in $[0, 1]$

$$\begin{aligned} \bullet \quad x_t &= \sum_{l=1}^{l_\varepsilon} \frac{(2i\pi t s(p))^l - t(2i\pi)^l s(p^l)}{l!} \text{ in } A; \\ \bullet \quad f_t : [0, 1] &\rightarrow A/J : \sigma \mapsto \sum_{l=1}^{l_\varepsilon} \frac{((2i\pi(1-\sigma)t + \sigma)p)^l - ((1-\sigma)t + \sigma)(2i\pi p)^l}{l!}. \end{aligned}$$

Then, $(1 + (y_t, f_t))_{t \in [0, 1]}$ is a path of $\alpha\varepsilon$ - $k_\varepsilon r$ unitary in C_q^+ with $x_0 = 0$ and $f_1 = 0$. Moreover,

- x_1 belongs to J and satisfies the conclusion of lemma 3.3 starting from the ε - r -projection p and with respect to the semi-split extension of filtered C^* -algebras $0 \rightarrow J \rightarrow A \xrightarrow{q} A/J \rightarrow 0$ and to the semi-split filtered cross-section s ;
- f_0 belongs to SA/J and satisfies the conclusion of lemma 3.3 starting from the ε - r -projection p and with respect to the semi-split extension of filtered C^* -algebras $0 \rightarrow SA/J \rightarrow CA/J \rightarrow A/J \rightarrow 0$ corresponding to evaluation at 1 and to the semi-split filtered cross-section $A/J \mapsto CA/J; a \mapsto [t \mapsto ta]$.

Hence, following the construction of proposition 3.4 in the even case, we obtain that $e_{q,*} \circ \mathcal{D}_{J,A}$ and $\phi_{q,*} \circ \mathcal{D}_{A/J}$ coincide on $\mathcal{K}_0(A/J)$.

Let us check now the odd case. Let u be an ε - r -unitary in $M_n(A/J)$. Pick any ε - r -unitary in some $M_j(A/J)$ such that $\text{diag}(u, v)$ is homotopic to I_{n+j} in $U_{n+j}^{3\varepsilon, 2r}(A/J)$. According to lemma 3.2, and up to replace v by $\text{diag}(v, I_k)$ for some

integer k , there exists an element w in $U_{n+j}^{3\alpha_e\varepsilon, 2k_e, 3\varepsilon r}(A)$ homotopic to I_{n+j} as a $3\alpha_e\varepsilon - 2k_e, 3\varepsilon r$ -unitary and such that $\|q(w) - \text{diag}(u, v)\| \leq 3\alpha_e\varepsilon$. Let $(w_t)_{t \in [0,1]}$ be a path in $U_{n+j}^{3\alpha_e\varepsilon, 2k_e, 3\varepsilon r}(A)$ with $w_0 = I_{n+j}$ and $w_1 = w$ and set $y_t = q(w_t) \text{diag}(I_n, 0) q(w_t^*)$. As in the proof of proposition 3.4, we see that y_t is an element in $P_{n+j}^{12\alpha_e\varepsilon, 4k_e, 3\varepsilon r}(A/J)$ such that $\|y_1 - \text{diag}(I_n, 0)\| \leq 9\alpha_e\varepsilon$. Define

$$g : [0, 1] \rightarrow M_{n+j}(A/J); t \mapsto y_t - \text{diag}(I_n, 0) - t(y_1 - \text{diag}(I_n, 0)).$$

Then $g + \text{diag}(I_n, 0)$ is the element of $P_{n+j}^{12\alpha_e\varepsilon, 4k_e, 3\varepsilon r}(S^+A/J)$ that we get from u and v when we perform the construction of proposition 3.4 in the odd case with respect to the extension $0 \rightarrow SA/J \rightarrow CA/J \rightarrow A/J \rightarrow 0$. Let us set now $x_t = w_t \text{diag}(I_n, 0) w_t^*$ and $h_t = x_t - \text{diag}(I_n, 0) - ts \circ q(x_1 - \text{diag}(I_n, 0))$ for t in $[0, 1]$. Then $\text{diag}(I_n, 0) + h_t$ belongs to $P_{n+j}^{12\alpha_e\varepsilon, 4k_e, 3\varepsilon r}(A)$ and $\text{diag}(I_n, 0) + h_1$ is the element of $P_{n+j}^{12\alpha_e\varepsilon, 4k_e, 3\varepsilon r}(J)$ that we get from u and v when we perform the construction of proposition 3.4 in the odd case with respect to the extension $0 \rightarrow J \rightarrow A \xrightarrow{q} A/J \rightarrow 0$. Eventually, if we define

$$H_t : [0, 1] \rightarrow M_{n+j}(A/J); \sigma \mapsto g(1-\sigma)t + \sigma,$$

then $((h_t, H_t) + \text{diag}(I_n, 0))_{t \in [0,1]}$ is a homotopy in $P_{n+j}^{12\alpha_e\varepsilon, 4k_e, 3\varepsilon r}(C_q^+)$ between $((0, g) + \text{diag}(I_n, 0))$ and $((h_1, 0) + \text{diag}(I_n, 0))$. Thus we obtain the result in the odd case. \square

As a consequence, we get that the controlled suspension morphism is compatible with the controlled boundary maps.

Proposition 3.15. *There exists a control pair (λ, h) such that for any semi-split extension of filtered C^* -algebras $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$, the following diagrams are (λ, h) -commutative:*

$$\begin{array}{ccc} \mathcal{K}_0(A/J) & \xrightarrow{\mathcal{D}_{A/J}} & \mathcal{K}_1(SA/J) \\ \mathcal{D}_{J,A} \downarrow & & \downarrow \mathcal{D}_{SJ,SA} \\ \mathcal{K}_1(J) & \xrightarrow{\mathcal{D}_J} & \mathcal{K}_0(SJ) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{K}_1(A/J) & \xrightarrow{\mathcal{D}_{A/J}} & \mathcal{K}_0(SA/J) \\ \mathcal{D}_{J,A} \downarrow & & \downarrow \mathcal{D}_{SJ,SA} \\ \mathcal{K}_0(J) & \xrightarrow{\mathcal{D}_J} & \mathcal{K}_1(SJ) \end{array}$$

where \mathcal{D}_J and $\mathcal{D}_{A/J}$ stands respectively for the controlled suspension morphisms $\mathcal{D}_{SJ,CJ}$ and $\mathcal{D}_{SA/J,CA/J}$.

Proof. Let $q_s : SA \rightarrow SA/J$ the suspension of the homomorphism $q : A \rightarrow A/J$. Applying lemma 3.14 to the extensions $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ and $0 \rightarrow SJ \rightarrow SA \rightarrow SA/J \rightarrow 0$ and using the naturality of controlled boundary maps mentioned

in remark 3.5, we get

$$\begin{aligned}
e_{q_s,*} \circ \mathcal{D}_{SJ,SA} \circ \mathcal{D}_{A/J} &= \phi_{q_s,*} \circ \mathcal{D}_{SA/J} \circ \mathcal{D}_{A/J} \\
&= \mathcal{D}_{SC_q} \circ \phi_{q,*} \circ \mathcal{D}_{A/J} \\
&= \mathcal{D}_{SC_q} \circ e_{q,*} \circ \mathcal{D}_{J,A} \\
&= e_{q_s,*} \circ \mathcal{D}_J \circ \mathcal{D}_{J,A}
\end{aligned}$$

The proposition is then a consequence of lemma 3.13. \square

4. CONTROLLED BOTT PERIODICITY

The aim of this section is to prove that there exists a control pair (λ, h) such that given a filtered C^* -algebra A , then Bott periodicity $K_0(A) \xrightarrow{\cong} K_0(S^2 A)$ is induced in K -theory by a (λ, h) -isomorphism $\mathcal{K}_0(A) \rightarrow \mathcal{K}_0(S^2 A)$. As an application, we use the controlled boundary morphism of proposition 3.4 to close the controlled exact sequence of 3.11 into a six-term (λ, h) -exact sequence for some universal control pair (λ, h) . This will be achieved by using the full power of KK -theory.

4.1. Tensorization in KK -theory. Let A be a C^* -algebra and let B be a C^* -algebra filtered by $(B_r)_{r>0}$. Within all this section, we will assume for sake of simplicity that B_r is closed for every positive number r (which is the case for Roe algebras and crossed product algebras). Let us define $A \otimes B_r$ as the closure in the spatial tensor product $A \otimes B$ of the algebraic tensor product of A and B_r . Then the C^* -algebra $A \otimes B$ is filtered by $(A \otimes B_r)_{r>0}$. Moreover, if J is a semi-split ideal of A , i.e $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ is a semi-split extension of C^* algebras, then

$$0 \rightarrow J \otimes B \rightarrow A \otimes B \rightarrow A/J \otimes B \rightarrow 0$$

is a semi-split extension of filtered C^* -algebras. Recall from [11] that for C^* -algebras A_1, A_2 and D , G. Kasparov defined a tensorization map

$$\tau_D : KK_*(A_1, A_2) \rightarrow KK_*(A_1 \otimes D, A_2 \otimes D)$$

in the following way: let z be an element in $KK_*(A_1, A_2)$ represented by a K -cycle (π, T, \mathcal{E}) , where

- \mathcal{E} is a right A_2 -Hilbert module;
- π is a representation of A_1 into the algebra $L(\mathcal{E})$ of adjointable operators of \mathcal{E} ;
- T is a self-adjoint operator on \mathcal{E} satisfying the K -cycle conditions, i.e. $[T, \pi(a)], \pi(a)(T^2 - Id_{\mathcal{E}})$ are compact operators on \mathcal{E} for any a in A_1 .

Then $\tau_D(z) \in KK_*(A_1 \otimes D, A_2 \otimes D)$ is represented by the K -cycle $(\pi \otimes Id_D, T \otimes Id_D, \mathcal{E} \otimes D)$.

In what follows, we show that if A_1 and A_2 are C^* -algebras, if B is a filtered C^* -algebra and if z is an element in $KK_*(A_1, A_2)$, then the homomorphism $KK_*(A_1 \otimes B) \rightarrow KK_*(A_2 \otimes B)$ provided by left multiplication by $\tau_B(z)$ is induced by a controlled morphism. Moreover, we have some compatibility results with respect to Kasparov product. As an outcome, we obtain a controlled version of the Bott periodicity that induces in K -theory the Bott periodicity.

Proposition 4.1. *Let A_1 and A_2 be C^* -algebras, let B be a filtered C^* -algebra and let z be an element in $KK_1(A_1, A_2)$. Then there exists an (α_D, k_D) -controlled*

morphism

$$\mathcal{T}_B(z) = (\tau_B^{\varepsilon, r}(z))_{0 < \varepsilon < \frac{1}{4\alpha_D}, r > 0} : \mathcal{K}_*(A_1 \otimes B) \rightarrow \mathcal{K}_*(A_2 \otimes B)$$

of degree 1 inducing in K -theory the right multiplication by $\tau_B(z)$.

Proof. Recall that z can be indeed represented by a odd A_1 - A_2 - K -cycle $(\pi, T, \mathcal{H} \otimes A_2)$, where \mathcal{H} is a separable Hilbert space, π is a representation of A_1 in the algebra $\mathcal{L}(\mathcal{H} \otimes A_2)$ of adjointable operators of $\mathcal{H} \otimes A_2$ and T is a self-adjoint operator in $\mathcal{L}(\mathcal{H} \otimes A_2)$ satisfying the K -cycle conditions. Let us set $P_B = \frac{\mathcal{I}d_{\mathcal{H} \otimes A_2 \otimes B} + T \otimes Id_B}{2}$, $\pi_B = \pi \otimes Id_B$ and define the C^* -algebra

$$E^{(\pi, T)} = \{(x, y) \in A_1 \otimes B \bigoplus \mathcal{L}(\mathcal{H} \otimes A_2 \otimes B) \text{ such that } P_B \cdot \pi_B(x) \cdot P_B - y \in \mathcal{K}(\mathcal{H}) \otimes A_2 \otimes B\}.$$

Since P_B has no propagation, the C^* -algebra $E^{(\pi, T)}$ is filtered by $(E_r^{(\pi, T)})_{r > 0}$ with

$$E_r^{(\pi, T)} = \{(x, P_B \cdot \pi_B(x) \cdot P_B + y); x \in A_1 \otimes B_r \text{ and } y \in \mathcal{K}(\mathcal{H}) \otimes A_2 \otimes B_r\}.$$

The extension of filtered C^* -algebras

$$(3) \quad 0 \longrightarrow \mathcal{K}(\mathcal{H}) \otimes A_2 \otimes B \longrightarrow E^{(\pi, T)} \longrightarrow A_1 \otimes B \longrightarrow 0$$

is semi-split by the cross-section

$$s : A_1 \otimes B \rightarrow E^{(\pi, T)}; x \mapsto (x, P_B \cdot \pi_B(x) \cdot P_B).$$

Let us show that the associated controlled boundary (degree one) map

$$\mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes A_2 \otimes B, E^{(\pi, T)}} : \mathcal{K}_*(A_1 \otimes B) \rightarrow \mathcal{K}_*(\mathcal{K}(\mathcal{H}) \otimes A_2 \otimes B)$$

only depends on the class z of $(\pi, T, \mathcal{H} \otimes A_2)$ in $KK_1(A_1, A_2)$. Assume that $(\pi, T, \mathcal{H} \otimes A_2[0, 1])$ is a A_1 - $A_2[0, 1]$ - K -cycle providing a homotopy between two A_1 - A_2 - K -cycles $(\pi_0, T_0, \mathcal{H} \otimes A_2)$ and $(\pi_1, T_1, \mathcal{H} \otimes A_2)$. For $t \in [0, 1]$ we denote by

- $e_t : A_2[0, 1] \rightarrow A_2$ the evaluation at t ;
- $F_t \in \mathcal{L}(\mathcal{H} \otimes A_2)$ the fiber at t of an operator $F \in \mathcal{L}(\mathcal{H} \otimes A_2[0, 1])$;
- $\pi_t : A_1 \rightarrow \mathcal{L}(\mathcal{H} \otimes A_2)$ the representation induced by π at the fiber t ;
- $s_t : A_2 \otimes B \rightarrow E^{(\pi_t, T_t)}; x \mapsto (x, P_{t, B} \cdot \pi_{t, B}(x) \cdot P_{t, B})$ (with $P = \frac{T+1}{2}$);

Then the homomorphism $E^{(\pi, T)} \rightarrow E^{(\pi_t, T_t)}; (x, y) \mapsto (x, y_t)$ satisfies the conditions of remark 3.5 (with $s : A_2 \otimes B \rightarrow E^{(\pi, T)}; x \mapsto (x, P_B \cdot \pi_B(x) \cdot P_B)$ and $s_t : A_2 \otimes B \rightarrow E^{(\pi_t, T_t)}$) and thus we get that

$$(\mathcal{I}d_{\mathcal{K}(\mathcal{H})} \otimes e_t \otimes Id_B)_* \circ \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes A_1 \otimes B[0, 1], E^{(\pi, T)}} = \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes A_1 \otimes B, E^{(\pi_t, T_t)}},$$

and according to lemma 1.27, we deduce that

$$\mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes A_1 \otimes B_2, E^{(\pi_0, T_0)}} = \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes A_1 \otimes B, E^{(\pi_1, T_1)}}.$$

This shows that for a A_1 - A_2 - K -cycle $(\pi, T, \mathcal{H} \otimes A_2)$, then $\mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes A_1 \otimes B, E^{(\pi, T)}}$ depends only on the class z of $(\pi, T, \mathcal{H} \otimes A_2)$ in $KK_1(A_1, A_2)$. Finally we define

$$\mathcal{T}_B(z) = (\tau_B^{\varepsilon, r}(z))_{0 < \varepsilon < \frac{1}{4\alpha_D}} \stackrel{\text{def}}{=} \mathcal{M}_{A_2 \otimes B}^{-1} \circ \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes A_1 \otimes B, E^{(\pi, T)}},$$

where

- $(\pi, T, \mathcal{H} \otimes A_2)$ is any A_1 - A_2 - K -cycles representing z ;
- $\mathcal{M}_{A_2 \otimes B}$ is the Morita equivalence (see example 2.2).

The result then follows from the observation that up to the Morita equivalence

$$K_*(\mathcal{K}(\mathcal{H}) \otimes A_2 \otimes B) \xrightarrow{\cong} K_*(A_2 \otimes B),$$

the boundary $\partial_{\mathcal{K}(\mathcal{H}) \otimes A_1 \otimes B, E^{(\pi, T)}}$ corresponding to the exact sequence (3) is induced by right multiplication by $\tau_B(z)$. \square

Remark 4.2. Let B be a filtered C^* -algebra.

- (i) For any C^* -algebras A_1 and A_2 and any elements z and z' in $KK_1(A_1, A_2)$ then

$$\mathcal{T}_B(z + z') = \mathcal{T}_B(z) + \mathcal{T}_B(z').$$

- (ii) Let $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ be a semi-split extension of filtered C^* -algebras and let $[\partial_{J,A}]$ be the element of $KK_1(A/J, J)$ that implements the boundary map $\partial_{J,A}$. Then we have

$$\mathcal{T}_B([\partial_{J,A}]) = \mathcal{D}_{J \otimes B, A \otimes B}.$$

- (iii) For any C^* -algebras A_1 , A_2 and D and any K -cycle $(\pi, T, \mathcal{H} \otimes A_2)$ for $KK_1(A_1, A_2)$, we have a natural identification between $E^{(\pi_D, T_D)}$ and $E^{(\pi, T)} \otimes D$. Hence, for any element z in $KK_1(A_1, A_2)$ then $\mathcal{T}_B(\tau_D(z)) = \mathcal{T}_{B \otimes D}(z)$.

For a filtered C^* -algebra B and a homomorphism $f : A_1 \rightarrow A_2$ of C^* -algebras, we set $f_B : A_1 \otimes B \rightarrow A_2 \otimes B$ for the filtered homomorphism induced by f .

Proposition 4.3. Let B be a filtered C^* -algebra and let A_1 and A_2 be two C^* -algebras.

- (i) For any C^* -algebra A'_1 , any homomorphism of C^* -algebras $f : A_1 \rightarrow A'_1$ and any z in $KK_1(A'_1, A_2)$, we have $\mathcal{T}_B(f^*(z)) = \mathcal{T}_B(z) \circ f_{B,*}$;
(ii) For any C^* -algebra A'_2 , any homomorphism of C^* -algebras $g : A_2 \rightarrow A'_2$ and any z in $KK_1(A_1, A_2)$, we have $\mathcal{T}_B(g_*(z)) = g_{B,*} \circ \mathcal{T}_B(z)$.

Proof.

- (i) Let A'_1 be a filtered C^* -algebra, let $f : A_1 \rightarrow A'_1$ be a homomorphism of C^* -algebras and let $(\pi, T, H \otimes A_2)$ be an odd A'_1 - A_2 - K -cycle. With the notations of the proof of proposition 4.1, the homomorphism

$$f^E : E^{f^*(\pi, T)} \rightarrow E^{(\pi, T)}; (x, y) \mapsto (f_B(x), y)$$

fits in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}(\mathcal{H}) \otimes A_2 \otimes B & \longrightarrow & E^{f^*(\pi, T)} & \longrightarrow & A_1 \otimes B \longrightarrow 0 \\ & & \downarrow & & f^E \downarrow & & \downarrow f_B \\ 0 & \longrightarrow & \mathcal{K}(\mathcal{H}) \otimes A_2 \otimes B & \longrightarrow & E^{(\pi, T)} & \longrightarrow & A'_1 \otimes B \longrightarrow 0 \end{array}.$$

Moreover f_B and f^E intertwines the semi-split and filtered cross-sections

$$A_1 \otimes B \rightarrow E^{f^*(\pi, T)}; x \mapsto (x, P_B \cdot \pi_B \circ f_B(x) \cdot P_B)$$

and

$$A'_1 \otimes B \rightarrow E^{(\pi, T)}; x \mapsto (x, P_B \cdot \pi_B(x) \cdot P_B)$$

and thus, we get by remark 3.5 that

$$\mathcal{T}_B(f^*(z)) = \mathcal{T}_B(z) \circ f_*$$

for all z in $KK_1(A'_1, A_2)$.

- (ii) Let A'_2 be a C^* -algebra and let $g : A_2 \rightarrow A'_2$ be a homomorphism of C^* -algebras. For any element F in $\mathcal{L}(\mathcal{H} \otimes A_2)$, let us denote by

$$\tilde{F} = F \otimes_{A_2} Id_{A'_2} \in \mathcal{L}(\mathcal{H} \otimes A_2 \otimes_{A_2} A'_2).$$

Notice that $\mathcal{H} \otimes A_2 \otimes_{A_2} A'_2$ can be viewed as a right A'_2 -Hilbert-submodule of $\mathcal{H} \otimes A'_2$ and under this identification, for any F in $\mathcal{K}(\mathcal{H}) \otimes A_2$, then \tilde{F} is the restriction to $\mathcal{H} \otimes A_2 \otimes_{A_2} A'_2$ of the homomorphism $(Id_{\mathcal{K}(\mathcal{H})} \otimes g)(F)$. Let z be an element of $KK_1(A_1, A_2)$ represented by a K -cycle $(\pi, T, \mathcal{H} \otimes A_2)$. Consider the A_1 - A_2 - K -cycle $(\pi', T', \mathcal{H}' \otimes A_2)$ with $\mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 are three copies of \mathcal{H} , $\pi' = 0 \oplus 0 \oplus \pi$ and $T' = Id_{\mathcal{H}_1 \otimes A_2} \oplus Id_{\mathcal{H}_2 \otimes A_2} \oplus T$. Then $(\pi', T', \mathcal{H}' \otimes A_2)$ is again a K -cycle representing z and $g_*(z)$ is represented by the K -cycle $(\pi'', T'', \mathcal{E})$, where

- $\mathcal{E} = \mathcal{H}_1 \otimes A'_2 \oplus \mathcal{H}_2 \otimes A'_2 \oplus \mathcal{H}_3 \otimes A_2 \otimes_{A_2} A'_2$;
- $\pi'' = 0 \oplus 0 \oplus \tilde{\pi}$;
- $T'' = Id_{\mathcal{H}_1 \otimes A'_2} \oplus Id_{\mathcal{H}_2 \otimes A'_2} \oplus \tilde{T}$.

Using Kasparov stabilization theorem, we get that $\mathcal{H}_2 \otimes A'_2 \oplus \mathcal{H}_3 \otimes A_2 \otimes_{A_2} A'_2$ is isomorphic as a right- A'_2 -Hilbert module to $\mathcal{H} \otimes A'_2$ and hence, using this identification, we can represent $g_*(z)$ using a standard right- A'_2 -Hilbert module, as in the proof of proposition 4.1. Then, under the above identification $\mathcal{H}_2 \otimes A'_2 \oplus \mathcal{H}_3 \otimes A_2 \otimes_{A_2} A'_2 \cong \mathcal{H} \otimes A'_2$,

$$\begin{aligned} g_E : E^{(\pi, T)} &\rightarrow E^{g_*(\pi, T)} \\ (x, y) &\mapsto (x, P_B'' \pi''(x) P_B'' + (Id_{\mathcal{K}(\mathcal{H}') \otimes B} \otimes g)(y - P_B' \pi'(x) P_B')) \end{aligned}$$

restricts to a homomorphism $\mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3) \otimes A_2 \otimes B \rightarrow \mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}) \otimes A'_2 \otimes B$.

We get now a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3) \otimes A_2 \otimes B & \longrightarrow & E^{(\pi', T')} & \longrightarrow & A_1 \otimes B \longrightarrow 0 \\ & & \downarrow g_E & & \downarrow g_E & & \downarrow = \\ 0 & \longrightarrow & \mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}) \otimes A'_2 \otimes B & \longrightarrow & E^{(\pi'', T'')} & \longrightarrow & A_1 \otimes B \longrightarrow 0 \end{array}$$

Hence, we get by remark 3.5 that

$$\mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes A'_2 \otimes B, E^{(\pi'', T'')}} = g_{E,*} \circ \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes A_2 \otimes B, E^{(\pi', T')}}.$$

But the restriction of g_E to the corner $\mathcal{K}(\mathcal{H}_1) \otimes A_2 \otimes B$ of the C^* -algebra $\mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3) \otimes A_2 \otimes B$ is $Id_{\mathcal{K}(\mathcal{H}_1)} \otimes g \otimes Id_B$. Since the Morita equivalence

$$\mathcal{M}_{A'_2 \otimes B} : \mathcal{K}_*(A'_2 \otimes B) \xrightarrow{\cong} \mathcal{K}_*(\mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}) \otimes A'_2 \otimes B)$$

can be implemented by an inclusion of $A'_2 \otimes B$ in a corner of $\mathcal{K}(\mathcal{H}_1) \otimes A'_2 \otimes B$, and similarly for the Morita equivalence

$$\mathcal{M}_{A_2 \otimes B} : \mathcal{K}_*(A_2 \otimes B) \xrightarrow{\cong} \mathcal{K}_*(\mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3) \otimes A_2 \otimes B),$$

we deduce that the two following compositions coincide:

$$\mathcal{K}_*(A_2 \otimes B) \xrightarrow{g_{B,*}} \mathcal{K}_*(A'_2 \otimes B) \xrightarrow{\mathcal{M}_{A'_2 \otimes B}} \mathcal{K}_*(\mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}) \otimes (A'_2 \otimes B))$$

and

$$\begin{aligned} \mathcal{K}_*(A_2 \otimes B) &\xrightarrow{\mathcal{M}_{A_2 \otimes B}} \mathcal{K}_*(\mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3) \otimes A_2 \otimes B) \\ &\xrightarrow{g_{E,*}} \mathcal{K}_*(\mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}) \otimes A'_2 \otimes B). \end{aligned}$$

Hence we get

$$\mathcal{T}_B(g_*(z)) = g_* \circ \mathcal{T}_B(z)$$

for any z in $KK_1(A_1, A_2)$.

□

Let us now extend the definition of \mathcal{T}_B to the even case. Consider for a suitable control pair (α_B, k_B) and any filtered C^* -algebra A the (α_B, k_B) -controlled morphism of odd degree $\mathcal{B}_A : \mathcal{K}_*(SA) \rightarrow \mathcal{K}_*(A)$ defined

- by \mathcal{B}_A^0 on $\mathcal{K}_0(SA)$ as in corollary 3.12;
- by $\mathcal{M}_A^{-1} \circ \mathcal{D}_{\mathcal{K}(\ell^2(\mathbb{N})) \otimes A, \tau_0 \otimes A}$ on $\mathcal{K}_1(SA)$ using the Toeplitz extension

$$0 \rightarrow \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \rightarrow \mathcal{T}_0 \otimes A \rightarrow SA \rightarrow 0$$

(see the discussion at the end of section 3.2).

Then, according to corollary 3.12 and proposition 3.7, there exists a control pair (λ, h) such that \mathcal{B}_A is a right (λ, h) -inverse for $\mathcal{D}_{SA, CA}$ for any filtered C^* -algebra A . Let us set $\alpha_{\mathcal{T}} = \lambda \alpha_B$ and $k_{\mathcal{T}} = h * k_B$.

Now, let B be a filtered C^* -algebra, let A_1 and A_2 be C^* -algebras, then define for any z in $KK_0(A_1, A_2)$ the $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ -controlled morphism

$$\mathcal{T}_B(z) = (\tau_B^{\varepsilon, r})_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{T}}}, r > 0} : \mathcal{K}_*(A_1 \otimes B) \rightarrow \mathcal{K}_*(A_2 \otimes B)$$

by

$$\mathcal{T}_B(z) \stackrel{\text{def}}{=} \mathcal{B}_{A_2 \otimes B} \circ \mathcal{T}_B(z \otimes_{A_2} [\partial_{A_2}])$$

where

- $[\partial_{A_2}] = [\partial_{SA_2, CA_2}] \in KK_1(A_2, SA_2)$ corresponds to the boundary of the exact sequence $0 \rightarrow SA_2 \rightarrow CA_2 \rightarrow A \rightarrow 0$;
- \otimes_{A_2} stands for Kasparov product.

Up to compose on the left with $\iota_*^{\alpha_{\mathcal{D}\varepsilon}, \alpha_{\mathcal{T}\varepsilon}, k_{\mathcal{D}r}, k_{\mathcal{T}r}}$, we can in the odd case define $\mathcal{T}_B(\bullet)$ also as an $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ -controlled morphism.

Theorem 4.4. *Let B be a filtered C^* -algebra, let A_1 and A_2 be C^* -algebras*

- For any element z in $KK_*(A_1, A_2)$, then $\mathcal{T}_B(z) : \mathcal{K}_*(A_1 \otimes B) \rightarrow \mathcal{K}_*(A_2 \otimes B)$ is a $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ -controlled morphism with same degree as z that induces in K -theory right multiplication by $\tau_B(z)$.*
- For any elements z and z' in $KK_*(A_1, A_2)$ then*

$$\mathcal{T}_B(z + z') = \mathcal{T}_B(z) + \mathcal{T}_B(z').$$

- Let A'_1 be a filtered C^* -algebras and let $f : A_1 \rightarrow A'_1$ be a homomorphism of C^* -algebras, then $\mathcal{T}_B(f^*(z)) = \mathcal{T}_B(z) \circ f_{B,*}$ for all z in $KK_*(A'_1, A_2)$.*
- Let A'_2 be a C^* -algebra and let $g : A'_2 \rightarrow A_2$ be a homomorphism of C^* -algebras then $\mathcal{T}_B(g_*(z)) = g_{B,*} \circ \mathcal{T}_B(z)$ for any z in $KK_*(A_1, A'_2)$.*
- $\mathcal{T}_B([Id_{A_1}]) \stackrel{(\alpha_{\mathcal{T}}, k_{\mathcal{T}})}{\sim} \mathcal{I}_{\mathcal{K}_*(A_1 \otimes B)}$.*
- For any C^* -algebra D and any element z in $KK_*(A_1, A_2)$, we have $\mathcal{T}_B(\tau_D(z)) = \mathcal{T}_{B \otimes D}(z)$.*

Proof. Since $\mathcal{B}_{A_2 \otimes B}$ is a right (λ, h) -inverse for $\mathcal{D}_{SA_2 \otimes B, CA_2 \otimes B}$, it induces in K -theory a right inverse (indeed an inverse) for the (degree 1) boundary map

$$\partial_{SA_2 \otimes B, CA_2 \otimes B} : \mathcal{K}_*(A_2 \otimes B) \rightarrow \mathcal{K}_*(SA_2 \otimes B).$$

But since $\mathcal{T}_B(z \otimes_{A_2} [\partial_{SA_2 \otimes B, CA_2 \otimes B}])$ induces in K -theory right multiplication by $z \otimes_{A_2} [\partial_{SA_2 \otimes B, CA_2 \otimes B}]$, we eventually get that $\mathcal{T}_B(z \otimes_{A_2} [\partial_{SA_2 \otimes B, CA_2 \otimes B}])$ induced in K -theory the composition

$$K_*(A_1 \otimes B) \xrightarrow{\otimes_{A_1 \otimes B} \mathcal{T}_B(z)} K_*(A_2 \otimes B) \xrightarrow{\partial_{SA_2 \otimes B, CA_2 \otimes B}} K_*(SA_2 \otimes B)$$

and hence we get the first point.

Point (ii) is a consequence of remark 4.2. Point (iii) is a consequence of proposition 4.3. Point (iv) is a consequence of proposition 4.3 and of the naturality of \mathcal{B}_\bullet (see remark 3.5 and corollary 3.12), point (v) holds by definition of \mathcal{B}_\bullet . Point (vi) is a consequence of point (iii) of remark 4.2. \square

We end this section by proving the compatibility of \mathcal{T}_B with Kasparov product.

Theorem 4.5. *There exists a control pair (λ, h) such that the following holds :*

let A_1, A_2 and A_3 be C^ -algebras and let B be a filtered C^* -algebra. Then for any z in $KK_*(A_1, A_2)$ and any z' in $KK_*(A_2, A_3)$, we have*

$$\mathcal{T}_B(z \otimes_{A_2} z') \stackrel{(\lambda, h)}{\sim} \mathcal{T}_B(z') \circ \mathcal{T}_B(z).$$

Proof. We first deal with the case z even. According to [12, Lemma 1.6.9], there exists a C^* -algebra A_4 and homomorphisms $\theta : A_4 \rightarrow A_1$ and $\eta : A_4 \rightarrow A_2$ such that

- the element $[\theta]$ of $KK_*(A_4, A_1)$ induced by θ is invertible.
- $z = \eta_*([\theta]^{-1})$.

Since $\theta_*([\theta]^{-1}) = [Id_{A_1}]$ in $KK_*(A_1, A_1)$, we get in view of remark 2.5 and of points (iii), (iv) and (v) of theorem 4.4 that

$$\mathcal{T}_B(z \otimes_{A_2} z') \stackrel{(\lambda, h)}{\sim} \mathcal{T}_B(\theta^*(z \otimes_{A_2} z')) \circ \mathcal{T}_B([\theta]^{-1}),$$

with $(\lambda, h) = (\alpha_{\mathcal{T}}^2, k_{\mathcal{T}} * k_{\mathcal{T}})$. But by bi-functoriality of KK -theory, we have $\theta^*(z \otimes_{A_2} z') = \eta^*(z')$ and then the result is a consequence of points (iii) and (iv) of theorem 4.4. We can proceed similarly when z' is even. Let us prove now the result when z and z' are odd. Then $[\partial_{A_2}] = [\partial_{SA_2, CA_2}]$ is an invertible element in $KK_1(A_2, SA_2)$ and $z \otimes_{A_2} z' = z \otimes_{A_2} [\partial_{A_2}] \otimes_{SA_2} [\partial_{A_2}]^{-1} \otimes_{A_2} z'$ and hence using the even case, we get that

$$(4) \quad \mathcal{T}_B(z \otimes_{A_2} z') \stackrel{(\lambda, h)}{\sim} \mathcal{T}_B([\partial_{A_2}]^{-1} \otimes_{A_2} z') \circ \mathcal{T}_B(z \otimes_{A_2} [\partial_{A_2}]).$$

But

$$(5) \quad \begin{aligned} \mathcal{T}_B([\partial_{A_2}]^{-1} \otimes_{A_2} z') &= \mathcal{B}_{A_3 \otimes B} \circ \mathcal{T}_B([\partial_{A_2}]^{-1} \otimes_{A_2} z' \otimes_{A_3} [\partial_{A_3}]) \\ &\stackrel{(\lambda', h')}{\sim} \mathcal{B}_{A_3 \otimes B} \circ \mathcal{T}_B(z' \otimes_{A_3} [\partial_{A_3}]) \circ \mathcal{T}_B([\partial_{A_2}]^{-1}) \end{aligned}$$

for some control pair (λ', h') , depending only on (λ, h) and $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$, where equation (5) holds by the even case applied to $z' \otimes_{A_3} [\partial_{A_3}]$ and $[\partial_{A_2}]^{-1}$. Hence, for a control pair (λ'', h'') -depending only on (λ, h) , we get applying the even case to $[\partial_{A_2}]^{-1}$ and $z \otimes_{A_2} [\partial_{A_2}]$ that

$$(6) \quad \mathcal{T}_B(z \otimes_{A_2} z') \stackrel{(\lambda'', h'')}{\sim} \mathcal{B}_{A_3 \otimes B} \circ \mathcal{T}_B(z' \otimes_{A_3} [\partial_{A_3}]) \circ \mathcal{T}_B(z).$$

In view of this equation, we deduce the odd case from the controlled Bott periodicity, which will be proved in the next lemma: if we set $[\partial] = [\partial_{C_0(0,1), C_0(0,1)}] \in KK_1(\mathbb{C}, C_0(0,1))$, then there exists a controlled (α, k) such that $\mathcal{T}_A([\partial]^{-1})$ is an

(α, k) -inverse for \mathcal{D}_A for any filtered C^* -algebra A . Indeed, from this claim and since for some control pair (α', k') , the $(\alpha_{\mathcal{B}}, k_{\mathcal{B}})$ -controlled morphism \mathcal{B}_A is for every filtered C^* -algebra A a right (α', k') -inverse for $\mathcal{T}_A([\partial])$, we get that

$$\mathcal{T}_A([\partial]^{-1}) \stackrel{(\alpha'', k'')}{\sim} \mathcal{B}_A$$

for some controlled pair (α'', k'') depending only on (α', k') and $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$. Noticing by using point (vi) of theorem 4.4, that $\mathcal{T}_{A_3 \otimes B}([\partial]^{-1}) = \mathcal{T}_B([\partial_{A_3}]^{-1})$, the proof of the theorem in the odd case is then by equation (6) a consequence of the even case applied to $[\partial_{A_3}]^{-1}$ and $z' \otimes_{A_3} [\partial_{A_3}]$ \square

4.2. The controlled Bott isomorphism. We prove in this subsection a controlled version of Bott periodicity. The proof use the even case of theorem 4.5 and is needed for the proof of the odd case. Let $A = (A_r)_{r>0}$ be a filtered C^* -algebra and let us assume that A_r is closed for every positive number r . Let us denote for short as before $\mathcal{D}_{SA, CA}$ by \mathcal{D}_A and $[\partial_{SA, CA}]$ by $[\partial_A]$ for any filtered C^* -algebra A and let us set $[\partial] = [\partial_{\mathbb{C}}]$.

Theorem 4.6. *There exists a control pair (α, k) such that for every filtered C^* -algebra A , then $\mathcal{T}_A([\partial]^{-1})$ is an (α, k) -inverse for \mathcal{D}_A .*

Proof. Consider the even element $z = [\partial] \otimes_S [\partial_S]$ of $KK_*(\mathbb{C}, S^2)$, where $S = C_0(0, 1)$ and $S^2 = SS$. The lemma is a consequence of the following claim: there exists a control pair (λ, h) such that $\mathcal{D}_{SA} \circ \mathcal{D}_A \stackrel{(\lambda, h)}{\sim} \mathcal{T}_A(z)$ for any C^* -algebra A . Before proving the claim, let us see how it implies the lemma. Notice first that by point (ii) of remark 4.2, we have $\mathcal{D}_A = \mathcal{T}_A([\partial])$. Since by associativity of Kasparov product $[\partial]^{-1} \otimes_{\mathbb{C}} z = [\partial_S]$, we get from theorem 4.5 applied to the even case, that there exists a control pair (λ', h') such that for any filtered C^* -algebra A , then $\mathcal{T}_A(z) \circ \mathcal{T}_A([\partial]^{-1}) \circ \mathcal{D}_A \stackrel{(\lambda, h)}{\sim} \mathcal{D}_{SA} \circ \mathcal{D}_A$. Using the claim and since z is an invertible element of $KK_*(\mathbb{C}, S^2)$, we obtain from theorem 4.5 applied to the even case that there exists a control pair (α, k) such that $\mathcal{T}_A([\partial]^{-1})$ is a left (α, k) -inverse for \mathcal{D}_A . Using associativity of the Kasparov product, we see that $[\partial] = z \otimes_{S^2} [\partial_S]^{-1}$. Then applying twice theorem 4.5, on one hand to $[\partial]^{-1}$ and $z \otimes_{S^2} [\partial_S]^{-1}$ and on the other hand to $[\partial]^{-1} \otimes z$ and $[\partial_S]^{-1}$, we get that there exists a control pair (α', k') such that $\mathcal{T}_A([\partial]) \circ \mathcal{T}_A([\partial]^{-1}) \stackrel{(\alpha', k')}{\sim} \mathcal{T}_{SA}([\partial]^{-1}) \circ \mathcal{T}_{SA}([\partial])$. But according to what we have seen before, $\mathcal{T}_{SA}([\partial]^{-1}) \circ \mathcal{T}_{SA}([\partial]) \stackrel{(\alpha, k)}{\sim} \mathcal{Id}_{\mathcal{K}_*(SA)}$.

Let us now prove the claim. It is known that up to Morita equivalence, $[\partial_A]^{-1}$ is the element of $KK_1(SA, A)$ corresponding to the boundary element of the Toeplitz extension

$$0 \rightarrow \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \rightarrow \mathcal{T}_0 \otimes A \rightarrow SA \rightarrow 0.$$

Let us respectively denote by $\mathcal{D}_A^0 : \mathcal{K}_0(A) \rightarrow \mathcal{K}_1(SA)$ and $\mathcal{D}_A^1 : \mathcal{K}_1(A) \rightarrow \mathcal{K}_0(SA)$ the restriction of \mathcal{D}_A to $\mathcal{K}_0(A)$ and $\mathcal{K}_1(A)$. According to proposition 3.7, there exists a control pair (λ', h') such that, on even elements

$$(7) \quad \mathcal{T}_A([\partial]^{-1}) \circ \mathcal{D}_A^0 \stackrel{(\lambda', h')}{\sim} \mathcal{Id}_{\mathcal{K}_0(A)}.$$

Since $[\partial_S] = [\partial]^{-1} \otimes z$, we get by left composition by $\mathcal{T}_A(z)$ in equation (7) and by using theorem 4.5 in the even case that there exists a control pair (λ, h) depending only on (λ', h') and such that that $\mathcal{D}_{SA}^1 \circ \mathcal{D}_A^0 \stackrel{(\lambda, h)}{\sim} \mathcal{T}_A^0(z)$ (here $\mathcal{T}_A^0(z) : \mathcal{K}_0(A) \rightarrow \mathcal{K}_0(S^2 A)$)

stands for the restriction of $\mathcal{T}_A(z)$ to $\mathcal{K}_0(A)$). For the odd case, we know from corollary 3.12 that there exists a control pair (λ'', h'') such that $\mathcal{D}_{S^2A}^1 : \mathcal{K}_1(S^2A) \rightarrow \mathcal{K}_0(S^3A)$ is (λ'', h'') -invertible. Using the previous case, and since by associativity of the Kasparov product, we have $[\partial_A] \otimes_{SA} \tau_{SA}(z) = \tau_A(z) \otimes [\partial_{S^2A}]$, we get by applying twice theorem 4.5 in the even case that there exists a control pair (λ''', h''') such that $\mathcal{D}_{S^2A}^1 \circ \mathcal{D}_{SA}^0 \circ \mathcal{D}_A^1 \stackrel{(\lambda''', h''')}{\sim} \mathcal{D}_{S^2A}^1 \circ \mathcal{T}_A^1(z)$, where $\mathcal{T}_A^1(z) : \mathcal{K}_1(A) \rightarrow \mathcal{K}_1(S^2A)$ is the restriction of $\mathcal{T}_A(z)$ to $\mathcal{K}_1(A)$. Since $\mathcal{D}_{S^2A}^1 : \mathcal{K}_1(S^2A) \rightarrow \mathcal{K}_0(S^3A)$ is (λ'', h'') -invertible, we get the result by remark 2.5. \square

4.3. The six term (λ, h) -exact sequence. Recall from proposition 3.15 that there exists a control pair (λ, h) such that for any semi-split extension of filtered C^* -algebras $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$, the following diagrams are (λ, h) -commutative:

$$\begin{array}{ccc} \mathcal{K}_0(A/J) & \xrightarrow{\mathcal{D}_{A/J}} & \mathcal{K}_1(SA/J) \\ \mathcal{D}_{J,A} \downarrow & & \downarrow \mathcal{D}_{SJ,SA} \\ \mathcal{K}_1(J) & \xrightarrow{\mathcal{D}_J} & \mathcal{K}_0(SJ) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{K}_1(A/J) & \xrightarrow{\mathcal{D}_{A/J}} & \mathcal{K}_0(SA/J) \\ \mathcal{D}_{J,A} \downarrow & & \downarrow \mathcal{D}_{SJ,SA} \\ \mathcal{K}_0(J) & \xrightarrow{\mathcal{D}_J} & \mathcal{K}_1(SJ) \end{array}$$

As a consequence, by using theorem 4.6 and proposition 3.11, we get

Theorem 4.7. *There exists a control pair (λ, h) such that for any semi-split extension of filtered C^* -algebras*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} A/J \longrightarrow 0,$$

with A_r closed for every positive number r , then the following six-term sequence is (λ, h) -exact

$$\begin{array}{ccccc} \mathcal{K}_0(J) & \xrightarrow{j_*} & \mathcal{K}_0(A) & \xrightarrow{q_*} & \mathcal{K}_0(A/J) \\ \mathcal{D}_{J,A} \uparrow & & & & \downarrow \mathcal{D}_{J,A} \\ \mathcal{K}_1(A/J) & \xleftarrow{q_*} & \mathcal{K}_1(A) & \xleftarrow{j_*} & \mathcal{K}_1(J) \end{array}$$

Remark 4.8. *Let us consider with notations of section 3.4 the semi-split extension of filtered C^* -algebras*

$$(8) \quad 0 \rightarrow SA/J \xrightarrow{\phi_q} C_q \xrightarrow{\pi_1} A \rightarrow 0,$$

where $\pi_1 : C_q \rightarrow A$ is the projection on the first factor of C_q . Since we have a semi-split extension of filtered algebras $0 \rightarrow J \xrightarrow{e_j} C_q \xrightarrow{\pi_2} A/J[0,1] \rightarrow 0$, and since $A/J[0,1]$ is a contractible filtered C^* -algebra, we see in view of theorem 4.7 that $e_{j,*} : \mathcal{K}_*(J) \rightarrow \mathcal{K}_*(C_q)$ is a controlled isomorphism. It is then plain to check that up to the controlled isomorphism $e_{j,*}$ and $\mathcal{D}_{A/J} : \mathcal{K}_*(SA/J) \rightarrow \mathcal{K}_*(A/J)$, we get from the semi-split extension of filtered C^* -algebras of equation (8) (for a possibly different control pair) the controlled six-term exact sequence of theorem 4.7.

If we apply theorem 4.7 to a filtered and split extension, we get:

Corollary 4.9. *There exists a control pair (λ, h) such that for every split extension of filtered C^* -algebra $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$, with A_r closed for every positive number r and any filtered split cross-section $s : A/J \rightarrow A$, then*

$$\mathcal{K}_*(J) \oplus \mathcal{K}_*(A/J) \longrightarrow \mathcal{K}_*(A); (x, y) \mapsto j_*(x) + s_*(y)$$

is (λ, h) -invertible.

5. QUANTITATIVE K -THEORY FOR CROSSED PRODUCT C^* -ALGEBRAS

In this section, we study quantitative K -theory for crossed product C^* -algebras and discuss its applications to K -amenability.

Let Γ be a finitely generated group. A Γ - C^* -algebra is a separable C^* -algebra equipped with an action of Γ by automorphisms. Recall that the convolution algebra $C_c(\Gamma, A)$ of finitely supported A -valued functions on Γ admits two canonical C^* -completions, the reduced crossed product $A \rtimes_{red} \Gamma$ and the maximal crossed product $A \rtimes_{max} \Gamma$. Moreover, there is a canonical epimorphism $\lambda_{\Gamma, A} : A \rtimes_{max} \Gamma \rightarrow A \rtimes_{red} \Gamma$ which is the identity on $C_c(\Gamma, A)$.

5.1. Lengths and propagation. Recall that a length on Γ is a map $\ell : \Gamma \rightarrow \mathbb{R}^+$ such that

- $\ell(\gamma) = 0$ if and only if γ is the identity element e of Γ ;
- $\ell(\gamma\gamma') \leq \ell(\gamma) + \ell(\gamma')$ for all element γ and γ' of Γ .
- $\ell(\gamma) = \ell(\gamma^{-1})$.

In what follows, we will assume that ℓ is a word length arising from a finite generating symmetric set S , i.e $\ell(\gamma) = \inf\{d \text{ such that } \gamma = \gamma_1 \cdots \gamma_d \text{ with } \gamma_1, \dots, \gamma_d \text{ in } S\}$. Let us denote by $B(e, r)$ the ball centered at the neutral element of Γ with radius r , i.e $B(e, r) = \{\gamma \in \Gamma \text{ such that } \ell(\gamma) \leq r\}$. For any positive number r , we set

$$(A \rtimes_{red} \Gamma)_r \stackrel{\text{def}}{=} \{f \in C_c(\Gamma, A) \text{ with support in } B(e, r)\}.$$

Then the C^* -algebra $A \rtimes_{red} \Gamma$ is filtered by $((A \rtimes_{red} \Gamma)_r)_{r>0}$. In the same way, setting $(A \rtimes_{max} \Gamma)_r \stackrel{\text{def}}{=} \{f \in C_c(\Gamma, A) \text{ with support in } B(e, r)\}$, then the C^* -algebra $A \rtimes_{max} \Gamma$ is filtered by $((A \rtimes_{max} \Gamma)_r)_{r>0}$ (notice that as sets, $(A \rtimes_{red} \Gamma)_r = (A \rtimes_{max} \Gamma)_r$). It is straightforward to check that two word lengths give rise for $A \rtimes_{red} \Gamma$ (resp. for $A \rtimes_{max} \Gamma$) to quantitative K -theories related by a $(1, c)$ -controlled isomorphism for a constant c .

For a homomorphism $f : A \rightarrow B$ of Γ - C^* -algebras, we denote respectively by $f_{\Gamma, red} : A \rtimes_{red} \Gamma \rightarrow B \rtimes_{red} \Gamma$ and $f_{\Gamma, max} : A \rtimes_{max} \Gamma \rightarrow B \rtimes_{max} \Gamma$ the homomorphisms respectively induced by f on the reduced and on the maximal crossed product.

For any semi-split extension of Γ - C^* -algebras $0 \rightarrow J \xrightarrow{j} A \xrightarrow{q} A/J \rightarrow 0$, we have semi-split extensions of filtered C^* -algebras

$$0 \rightarrow J \rtimes_{red} \Gamma \xrightarrow{j_{\Gamma, red}} A \rtimes_{red} \Gamma \xrightarrow{q_{\Gamma, red}} A/J \rtimes_{red} \Gamma \rightarrow 0$$

and

$$0 \rightarrow J \rtimes_{max} \Gamma \xrightarrow{j_{\Gamma, max}} A \rtimes_{max} \Gamma \xrightarrow{q_{\Gamma, max}} A/J \rtimes_{max} \Gamma \rightarrow 0$$

and hence, by theorem 4.7, we get:

Proposition 5.1. *There exists a control pair (λ, h) such that for any semi-split extension of Γ - C^* -algebras*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} A/J \longrightarrow 0,$$

the following six-term sequences are (λ, h) -exact

$$\begin{array}{ccccc} \mathcal{K}_0(J \rtimes_{red} \Gamma) & \xrightarrow{J_{\Gamma, red, *}} & \mathcal{K}_0(A \rtimes_{red} \Gamma) & \xrightarrow{q_{\Gamma, red, *}} & \mathcal{K}_0(A/J \rtimes_{red} \Gamma) \\ \mathcal{D}_{J \rtimes_{red} \Gamma, A \rtimes_{red} \Gamma} \uparrow & & & & \mathcal{D}_{J \rtimes_{red} \Gamma, A \rtimes_{red} \Gamma} \downarrow \\ \mathcal{K}_1(A/J \rtimes_{red} \Gamma) & \xleftarrow{q_{\Gamma, red, *}} & \mathcal{K}_1(A \rtimes_{red} \Gamma) & \xleftarrow{J_{\Gamma, red, *}} & \mathcal{K}_1(J \rtimes_{red} \Gamma) \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{K}_0(J \rtimes_{max} \Gamma) & \xrightarrow{J_{\Gamma, max, *}} & \mathcal{K}_0(A \rtimes_{max} \Gamma) & \xrightarrow{q_{\Gamma, max, *}} & \mathcal{K}_0(A/J \rtimes_{max} \Gamma) \\ \mathcal{D}_{J \rtimes_{red} \Gamma, A \rtimes_{max} \Gamma} \uparrow & & & & \mathcal{D}_{J \rtimes_{max} \Gamma, A \rtimes_{max} \Gamma} \downarrow \\ \mathcal{K}_1(A/J \rtimes_{max} \Gamma) & \xleftarrow{q_{\Gamma, max, *}} & \mathcal{K}_1(A \rtimes_{max} \Gamma) & \xleftarrow{J_{\Gamma, max, *}} & \mathcal{K}_1(J \rtimes_{max} \Gamma) \end{array}$$

5.2. Kasparov transformation. In this subsection we see how a slight modification of the argument used in section 4.1 allowed to define a controlled version of the Kasparov transformation compatible with Kasparov product.

Notice first that every element z of $KK_*^\Gamma(A, B)$ can be represented by a K -cycle, $(\pi, T, \mathcal{H} \otimes B)$, where

- \mathcal{H} is a separable Hilbert space;
- the right Hilbert B -module $\mathcal{H} \otimes B$ is acted upon by Γ ;
- π is an equivariant representation of A in the algebra $\mathcal{L}(\mathcal{H} \otimes B)$ of adjointable operators on $\mathcal{H} \otimes B$;
- T is a self-adjoint operator on $\mathcal{H} \otimes B$ satisfying the K -cycle conditions, i.e. $[T, \pi(a)], \pi(a)(T^2 - \mathcal{I}d_{\mathcal{H} \otimes B})$ and $\pi(a)(\gamma(T) - T)$ belongs to $\mathcal{K}(\mathcal{H}) \otimes B$, for every a in A and $\gamma \in \Gamma$.

Let $T_\Gamma = T \otimes_B \mathcal{I}d_{B \rtimes_{red} \Gamma}$ be the adjointable element of $(\mathcal{H} \otimes B) \otimes_B B \rtimes_{red} \Gamma \cong \mathcal{H} \otimes B \rtimes_{red} \Gamma$ induced by T and let π_Γ be the representation of $A \rtimes_{red} \Gamma$ in the algebra $\mathcal{L}(\mathcal{H} \otimes B \rtimes_{red} \Gamma)$ of adjointable operators of $\mathcal{H} \otimes B \rtimes_{red} \Gamma$ induced by π . Then $(\pi_\Gamma, T_\Gamma, \mathcal{H} \otimes B \rtimes_{red} \Gamma)$ is a $A \rtimes_{red} \Gamma$ - $B \rtimes_{red} \Gamma$ - K -cycle and the Kasparov transform [11] of z is the class $J_\Gamma^{red}(z)$ of this K -cycle in $KK_*(A \rtimes_{red} \Gamma, B \rtimes_{red} \Gamma)$. In the odd case, let us set $P = \frac{\mathcal{I}d_{\mathcal{H} \otimes B} + T}{2}$. Then P induces an adjointable operator $P_\Gamma = P \otimes_B \mathcal{I}d_{B \rtimes_{red} \Gamma}$ of $(\mathcal{H} \otimes B) \otimes_B B \rtimes_{red} \Gamma \cong \mathcal{H} \otimes B \rtimes_{red} \Gamma$. Let us define

$$E^{(\pi, T)} = \{(x, y) \in A \rtimes_{red} \Gamma \oplus \mathcal{L}(\mathcal{H} \otimes B \rtimes_{red} \Gamma) \text{ such that } P_\Gamma \cdot \pi_\Gamma(x) \cdot P_\Gamma - y \in \mathcal{K}(\mathcal{H}) \otimes B \rtimes_{red} \Gamma\}.$$

Since P_Γ has no propagation, the C^* -algebra $E^{(\pi, T)}$ is filtered by $(E_r^{(\pi, T)})_{r \geq 0}$ with

$$E_r^{(\pi, T)} = \{(x, P_\Gamma \cdot \pi_\Gamma(x) \cdot P_\Gamma + y); x \in (A \rtimes_{red} \Gamma)_r \text{ and } y \in \mathcal{K}(\mathcal{H}) \otimes (B \rtimes_{red} \Gamma)_r\}.$$

The extension of C^* -algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \otimes B \rtimes_{red} \Gamma \longrightarrow E^{(\pi, T)} \longrightarrow A \rtimes_{red} \Gamma \longrightarrow 0$$

is filtered semi-split by the cross-section

$$s : A \rtimes_{red} \Gamma \rightarrow E^{(\pi, T)}; x \mapsto (x, P_\Gamma \cdot \pi_\Gamma(x) \cdot P_\Gamma).$$

Let us show that $\mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes B \rtimes_{red} \Gamma, E^{(\pi, T)}}$ only depends on the class of $(\pi, T, \mathcal{H} \otimes B)$ in $KK_1^\Gamma(A, B)$. Assume that $(\pi, T, \mathcal{H} \otimes B[0, 1])$ is a Γ -equivariant A - $B[0, 1]$ - K -cycle

providing a homotopy between two Γ -equivariant A - B - K -cycles $(\pi_0, T_0, \mathcal{H} \otimes B)$ and $(\pi_1, T_1, \mathcal{H} \otimes B)$. For $t \in [0, 1]$ we denote by

- $e_t : B[0, 1] \rtimes_{red} \Gamma \rightarrow B \rtimes_{red} \Gamma$ the evaluation at t ;
- $F_t \in \mathcal{L}(\mathcal{H} \otimes B \rtimes_{red} \Gamma)$ the fiber at t of an operator $F \in \mathcal{L}(\mathcal{H} \otimes B[0, 1] \rtimes_{red} \Gamma)$;
- $\pi_{\Gamma, t}$ the representation of $A \rtimes_{red} \Gamma$ induced by π_{Γ} at the fiber t ;
- $s_t : A \rtimes_{red} \Gamma \rightarrow E^{(\pi_t, T_t)}$; $x \mapsto (x, P_{\Gamma, t} \cdot \pi_{\Gamma, t} \cdot P_{\Gamma, t})$ (with $P = \frac{T+1}{2}$);

Then the homomorphism $E^{(\pi, T)} \rightarrow E^{(\pi_t, T_t)}$; $(x, y) \mapsto (x, y_t)$ satisfies the conditions of remark 3.5 (with $s : A \rtimes_{red} \Gamma \rightarrow E^{(\pi, T)}$; $x \mapsto (x, P_{\Gamma} \cdot \pi_{\Gamma}(x) \cdot P_{\Gamma})$ and $s_t : A \rtimes_{red} \Gamma \rightarrow E^{(\pi_t, T_t)}$) and thus we get that

$$(\mathcal{I}d_{\mathcal{K}(\mathcal{H})} \otimes e_t)_* \circ \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes B[0, 1] \rtimes_{red} \Gamma, E^{(\pi, T)}} = \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes B \rtimes_{red} \Gamma, E^{(\pi_t, T_t)}},$$

and according to lemma 1.27, we deduce that

$$\mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes B \rtimes_{red} \Gamma, E^{(\pi_0, T_0)}} = \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes B \rtimes_{red} \Gamma, E^{(\pi_1, T_1)}}.$$

This shows that for a Γ -equivariant A - B - K -cycles $(\pi, T, \mathcal{H} \otimes B)$, then $\mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes B \rtimes_{red} \Gamma, E^{(\pi, T)}}$ depends only on the class z of $(\pi, T, \mathcal{H} \otimes B)$ in $KK_1^{\Gamma}(A, B)$. Eventually, if we define

$$\mathcal{J}_{\Gamma}^{red}(z) = \mathcal{M}_{B \rtimes_{red} \Gamma}^{-1} \circ \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes B \rtimes_{red} \Gamma, E^{(\pi, T)}},$$

where

- $(\pi, T, \mathcal{H} \otimes B)$ is any Γ -equivariant A - B - K -cycles representing z ;
- $\mathcal{M}_{B \rtimes_{red} \Gamma}$ is the Morita equivalence (see example 2.2).

we get as in section 4.1

Proposition 5.2. *Let A and B be Γ - C^* -algebras. Then for any element z of $KK_1^{\Gamma}(A, B)$, there is a odd degree $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ -controlled morphism*

$$\mathcal{J}_{\Gamma}^{red}(z) = (\mathcal{J}_{\Gamma}^{red, \varepsilon, r}(z))_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{D}}}, r > 0} : \mathcal{K}_*(A \rtimes_{red} \Gamma) \rightarrow \mathcal{K}_*(B \rtimes_{red} \Gamma)$$

such that

- $\mathcal{J}_{\Gamma}^{red}(x)$ induces in K -theory the right multiplication by $\mathcal{J}_{\Gamma}^{red}(z)$;
- $\mathcal{J}_{\Gamma}^{red}$ is additive, i.e

$$\mathcal{J}_{\Gamma}^{red}(z + z') = \mathcal{J}_{\Gamma}^{red}(z) + \mathcal{J}_{\Gamma}^{red}(z').$$

- Let A' be a Γ - C^* -algebra and let $f : A \rightarrow A'$ be a homomorphism Γ - C^* -algebras, then

$$\mathcal{J}_{\Gamma}^{red}(f^*(z)) = \mathcal{J}_{\Gamma}^{red}(z) \circ f_{\Gamma, red, *}$$

for any z in $KK_1^{\Gamma}(A', B)$.

- Let B' be a Γ - C^* -algebra and let $g : B \rightarrow B'$ be a homomorphism of Γ - C^* -algebras, then

$$\mathcal{J}_{\Gamma}^{red}(g_*(z)) = g_{\Gamma, red, *} \circ \mathcal{J}_{\Gamma}^{red}(z)$$

for any z in $KK_1^{\Gamma}(A, B)$.

- If

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$$

is a semi-split exact sequence of Γ - C^* -algebras, let $[\partial_{J, A}]$ be the element of $KK_1^{\Gamma}(A/J, J)$ that implements the boundary map $\partial_{J, A}$. Then we have

$$\mathcal{J}_{\Gamma}^{red}([\partial_{J, A}]) = \mathcal{D}_{J \rtimes_{red} \Gamma, A \rtimes_{red} \Gamma}.$$

We can now define $\mathcal{J}_\Gamma^{\text{red}}$ for even element in the following way. Set $\alpha_{\mathcal{J}} = \alpha_{\mathcal{T}}\alpha_{\mathcal{D}}$ and $k_{\mathcal{J}} = k_{\mathcal{T}}*k_{\mathcal{D}}$. If A and B are Γ - C^* -algebra and if z is an element in $KK_0^\Gamma(A, B)$, then we set with notation of section 4.1

$$\mathcal{J}_\Gamma^{\text{red}}(z) = (J_\Gamma^{\text{red}, \varepsilon, r}(z))_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{J}}}, r} \stackrel{\text{def}}{=} \mathcal{T}_{B \rtimes_{\text{red}} \Gamma}([\partial]^{-1}) \circ \mathcal{J}_\Gamma^{\text{red}}(z \otimes_B [\partial_{SB}]).$$

According to theorem 4.6, there exists a control pair (λ, h) such that for any Γ - C^* -algebra A , then $\mathcal{J}_\Gamma^{\text{red}}([Id_A]) \stackrel{(\lambda, h)}{\sim} \mathcal{I}d_{\mathcal{K}_*(A \rtimes_{\text{red}} \Gamma)}$. Up to compose with $\iota_*^{\alpha_{\mathcal{D}}\varepsilon, \alpha_{\mathcal{J}}\varepsilon, k_{\mathcal{D}}, \varepsilon r, k_{\mathcal{J}}, \varepsilon r}$, we can assume indeed that $\mathcal{J}_\Gamma^{\text{red}}(\bullet)$ is also, in the odd case a $(\alpha_{\mathcal{J}}, k_{\mathcal{J}})$ -controlled morphism. As for theorem 4.4, we get.

Theorem 5.3. *Let A and B be Γ - C^* -algebras.*

- (i) *For any element z of $KK_*^\Gamma(A, B)$, then*

$$\mathcal{J}_\Gamma^{\text{red}}(z) : \mathcal{K}_*(A \rtimes_{\text{red}} \Gamma) \rightarrow \mathcal{K}_*(B \rtimes_{\text{red}} \Gamma)$$

is a $(\alpha_{\mathcal{J}}, k_{\mathcal{J}})$ -controlled morphism of same degree as z that induces in K -theory right multiplication by $\mathcal{J}_\Gamma^{\text{red}}(z)$.

- (ii) *For any z and z' in $KK_*^\Gamma(A, B)$, then*

$$\mathcal{J}_\Gamma^{\text{red}}(z + z') = \mathcal{J}_\Gamma^{\text{red}}(z) + \mathcal{J}_\Gamma^{\text{red}}(z').$$

- (iii) *For any Γ - C^* -algebra A' , any homomorphism $f : A \rightarrow A'$ of Γ - C^* -algebras and any z in $KK_*^\Gamma(A', B)$, then $\mathcal{J}_\Gamma^{\text{red}}(f^*(z)) = \mathcal{J}_\Gamma^{\text{red}}(z) \circ f_{\Gamma, *}$.*

- (iv) *For any Γ - C^* -algebra B' , any homomorphism $g : B \rightarrow B'$ of Γ - C^* -algebras and any z in $KK_*^\Gamma(A, B)$, then $\mathcal{J}_\Gamma^{\text{red}}(g_*(z)) = g_{\Gamma, *} \circ \mathcal{J}_\Gamma^{\text{red}}(z)$.*

Using the same argument as in the proof of theorem 4.5, we see that $\mathcal{J}_\Gamma^{\text{red}}$ is compatible with Kasparov products.

Theorem 5.4. *There exists a control pair (λ, h) such that the following holds: for every Γ - C^* -algebras A, B and D , any elements z in $KK_*^\Gamma(A, B)$ and z' in $KK_*^\Gamma(B, D)$, then*

$$\mathcal{J}_\Gamma^{\text{red}}(z \otimes_B z') \stackrel{(\lambda, h)}{\sim} \mathcal{J}_\Gamma^{\text{red}}(z') \circ \mathcal{J}_\Gamma^{\text{red}}(z).$$

We can perform a similar construction for maximal cross products.

Theorem 5.5. *Let A and B be Γ - C^* -algebras.*

- (i) *For any element z of $KK_*^\Gamma(A, B)$, there exists a $(\alpha_{\mathcal{J}}, k_{\mathcal{J}})$ -controlled morphism*

$$\mathcal{J}_\Gamma^{\text{max}}(z) = (J_\Gamma^{\text{max}, \varepsilon, r}(z))_{0 < \varepsilon < \frac{1}{4\alpha_{\mathcal{J}}}, r} : \mathcal{K}_*(A \rtimes_{\text{max}} \Gamma) \rightarrow \mathcal{K}_*(B \rtimes_{\text{max}} \Gamma)$$

*with same degree as z that induces in K -theory right multiplication by $\mathcal{J}_\Gamma^{\text{max}}(z)$ and such that $\lambda_{\Gamma, B, *} \circ \mathcal{J}_\Gamma^{\text{max}}(z) = \mathcal{J}_\Gamma^{\text{red}}(z) \circ \lambda_{\Gamma, A, *}$.*

- (ii) *For any z and z' in $KK_*^\Gamma(A, B)$, then*

$$\mathcal{J}_\Gamma^{\text{max}}(z + z') = \mathcal{J}_\Gamma^{\text{max}}(z) + \mathcal{J}_\Gamma^{\text{max}}(z').$$

- (iii) *For any Γ - C^* -algebra A' , any homomorphism $f : A \rightarrow A'$ of Γ - C^* -algebras and any z in $KK_*^\Gamma(A', B)$, then $\mathcal{J}_\Gamma^{\text{max}}(f^*(z)) = \mathcal{J}_\Gamma^{\text{max}}(z) \circ f_{\Gamma, \text{max}, *}$.*

- (iv) *For any Γ - C^* -algebra B' , any homomorphism $g : B \rightarrow B'$ of Γ - C^* -algebras and any z in $KK_*^\Gamma(A, B)$, then $\mathcal{J}_\Gamma^{\text{max}}(g_*(z)) = g_{\Gamma, \text{max}, *} \circ \mathcal{J}_\Gamma^{\text{max}}(z)$.*

Moreover, there exists a controlled pair (λ, h) such that,

- *for any Γ algebra A , then $\mathcal{J}_\Gamma^{\text{max}}([Id_A]) \stackrel{(\lambda, h)}{\sim} \mathcal{I}d_{\mathcal{K}_*(A \rtimes_{\text{max}} \Gamma)}$;*

- For any semi-split extension of Γ algebras $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$, then $\mathcal{J}_\Gamma^{\max}([\partial_{J,A}]) \stackrel{(\lambda,h)}{\sim} \mathcal{D}_{J,A}$.

Theorem 5.6. *There exists a control pair (λ, h) such that the following holds: for every Γ - C^* -algebras A, B and D , any elements z in $KK_*^\Gamma(A, B)$ and z' in $KK_*^\Gamma(B, D)$, then*

$$\mathcal{J}_\Gamma^{\max}(z \otimes_B z') \stackrel{(\lambda,h)}{\sim} \mathcal{J}_\Gamma^{\max}(z') \circ \mathcal{J}_\Gamma^{\max}(z).$$

5.3. Application to K -amenability. The original definition of K -amenability is due to J. Cuntz [6]. For our purpose, it is more convenient to use the equivalent definition given by P. Julg and A. Valette in [10]. If Γ is a discrete group, let us denote by 1_Γ the class in $KK_0^\Gamma(\mathbb{C}, \mathbb{C})$ of the K -cycle $(Id_\mathbb{C}, 0, \mathbb{C})$, where \mathbb{C} is provided with the trivial action on Γ .

Definition 5.7. *Let Γ be a discrete group. Then Γ is K -amenable if 1_Γ can be represented by a K -cycle such that the action of Γ on the underlying Hilbert space is weakly contained in the regular representation.*

(The previous definition indeed also makes sense for locally compact groups.)

Example 5.8. *Amenable groups are obviously K -amenable. Typical example on non-amenable K -amenable groups are free groups [6]. More generally, J. L. Tu proved in [17] that group which satisfies the strong Baum-Connes conjecture (i.e with $\gamma = 1$) are K -amenable. Examples of such group are groups with the Haagerup property [8] and fundamental groups of compact and oriented 3-manifolds [13].*

For a Γ - C^* -algebra B and an element T of $L(\mathcal{H} \otimes B)$, where \mathcal{H} is a separable Hilbert space, let us set $T_{\Gamma, \max} = T \otimes_B Id_{B \rtimes_{\max} \Gamma}$ and $T_{\Gamma, \text{red}} = T \otimes_B Id_{B \rtimes_{\text{red}} \Gamma}$. If A is a Γ - C^* -algebra and $\pi : A \rightarrow L(\mathcal{H} \otimes B)$ is a Γ -equivariant representation, let $\pi_{\Gamma, \text{red}} : A \rtimes_{\text{red}} \Gamma \rightarrow L(\mathcal{H} \otimes B \rtimes_{\text{red}} \Gamma)$ and $\pi_{\Gamma, \max} : A \rtimes_{\max} \Gamma \rightarrow L(\mathcal{H} \otimes B \rtimes_{\max} \Gamma)$ be respectively the reduced and the maximal representation induced by π . Then, we have the following (compare with the proof of [10, proposition 3.4]).

Proposition 5.9. *Let Γ be a K -amenable discrete group and let A and B be Γ - C^* -algebras. Then any elements of $KK_*^\Gamma(A, B)$ can be represented by a K -cycle $(\pi, T, \mathcal{H} \otimes B)$ such that the homomorphism $\pi_{\Gamma, \max} : A \rtimes_{\max} \Gamma \rightarrow L(\mathcal{H} \otimes B \rtimes_{\max} \Gamma)$ factorises through the homomorphism $\lambda_{\Gamma, A} : A \rtimes_{\max} \Gamma \rightarrow A \rtimes_{\text{red}} \Gamma$, i.e there exists a homomorphism*

$$\pi_{\Gamma, \text{red}, \max} : A \rtimes_{\text{red}} \Gamma \rightarrow L(\mathcal{H} \otimes B \rtimes_{\max} \Gamma)$$

such that

$$\pi_{\Gamma, \max} = \pi_{\Gamma, \text{red}, \max} \circ \lambda_{\Gamma, A}.$$

As a consequence, for any Γ - C^* -algebra A , then

$$\lambda_{\Gamma, A, *} : K_*(A \rtimes_{\max} \Gamma) \rightarrow K_*(A \rtimes_{\text{red}} \Gamma)$$

is an isomorphism [6].

We have the following analogous result for quantitative K -theory.

Theorem 5.10. *There exists a control pair (λ, h) such that*

$$\lambda_{\Gamma, A, *} : \mathcal{K}_*(A \rtimes_{\max} \Gamma) \rightarrow \mathcal{K}_*(A \rtimes_{\text{red}} \Gamma)$$

is a (λ, h) -isomorphism for every Γ - C^* -algebra A .

Proof. Let $(\pi, T, \mathcal{H} \otimes SA)$ be a Γ -equivariant K -cycle as in proposition 5.9 representing the element $[\partial_A]$ of $KK_1^\Gamma(A, SA)$ corresponding to the extension

$$0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0.$$

Let then choose $\pi_{\Gamma, A, red, max} : A \rtimes_{red} \Gamma \rightarrow \mathcal{L}(\mathcal{H} \otimes B \rtimes_{max} \Gamma)$ such that $\pi_{\Gamma, max} = \pi_{\Gamma, red, max} \circ \lambda_{\Gamma, A}$. Let us set $P = \frac{T + Id_{\mathcal{H} \otimes SA}}{2}$ and then define

$$E_{red}^{(\pi, T)} = \{(x, y) \in A \rtimes_{red} \Gamma \oplus \mathcal{L}(\mathcal{H} \otimes SA \rtimes_{red} \Gamma) \text{ such that} \\ P_{\Gamma, red} \cdot \pi_{\Gamma, red}(x) \cdot P_{\Gamma, red} - y \in \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{red} \Gamma\},$$

$$E_{max}^{(\pi, T)} = \{(x, y) \in A \rtimes_{max} \Gamma \oplus \mathcal{L}(\mathcal{H} \otimes SA \rtimes_{max} \Gamma) \text{ such that} \\ P_{\Gamma, max} \cdot \pi_{\Gamma, max}(x) \cdot P_{\Gamma, max} - y \in \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma\}$$

and

$$E_{red, max}^{(\pi, T)} = \{(x, y) \in A \rtimes_{red} \Gamma \oplus \mathcal{L}(\mathcal{H} \otimes SA \rtimes_{max} \Gamma) \text{ such that} \\ P_{\Gamma, max} \cdot \pi_{\Gamma, red, max}(x) \cdot P_{\Gamma, max} - y \in \mathcal{K}(\mathcal{H}) \otimes A \rtimes_{max} \Gamma\}$$

Then $E_{red}^{(\pi, T)}$, $E_{max}^{(\pi, T)}$ and $E_{red, max}^{(\pi, T)}$ are respectively filtered by

$$\{(x, P_{\Gamma, red} \cdot \pi_{\Gamma, red}(x) \cdot P_{\Gamma, red} + y); x \in A \rtimes_{red} \Gamma_r \text{ and } y \in \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{red} \Gamma_r\}, \\ \{(x, P_{\Gamma, max} \cdot \pi_{\Gamma, max}(x) \cdot P_{\Gamma, max} + y); x \in SA \rtimes_{max} \Gamma_r \text{ and } y \in \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma_r\} \\ \text{and} \\ \{(x, P_{\Gamma, max} \cdot \pi_{\Gamma, red, max}(x) \cdot P_{\Gamma, max} + y); x \in A \rtimes_{red} \Gamma_r \text{ and } y \in \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma_r\}.$$

Moreover, the extension of C^* -algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{red} \Gamma \longrightarrow E_{red}^{(\pi, T)} \longrightarrow A \rtimes_{red} \Gamma \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma \longrightarrow E_{max}^{(\pi, T)} \longrightarrow A \rtimes_{max} \Gamma \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma \longrightarrow E_{red, max}^{(\pi, T)} \longrightarrow A \rtimes_{red} \Gamma \longrightarrow 0$$

provided by the projection on the first factor are respectively semi-split by the filtered cross-sections

$$s_{red} : A \rtimes_{red} \Gamma \rightarrow E_{red}^{(\pi, T)}; x \mapsto (x, P_{\Gamma, red} \cdot \pi_{\Gamma, red}(x) \cdot P_{\Gamma, red}),$$

$$s_{max} : A \rtimes_{max} \Gamma \rightarrow E_{max}^{(\pi, T)}; x \mapsto (x, P_{\Gamma, max} \cdot \pi_{\Gamma, max}(x) \cdot P_{\Gamma, max})$$

and

$$s_{red, max} : A \rtimes_{red} \Gamma \rightarrow E_{red, max}^{(\pi, T)}; x \mapsto (x, P_{\Gamma, max} \cdot \pi_{\Gamma, red, max}(x) \cdot P_{\Gamma, max}).$$

Let us set

$$f_1 : E_{max}^{(\pi, T)} \rightarrow E_{red, max}^{(\pi, T)} : (x, y) \mapsto (\lambda_{\Gamma, A, *}(x), y)$$

and

$$f_2 : E_{red, max}^{(\pi, T)} \rightarrow E_{red}^{(\pi, T)} : (x, y) \mapsto (x, y \otimes_{A \rtimes_{max} \Gamma} Id_{A \rtimes_{red} \Gamma}).$$

The the three above extensions fit in a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma & \longrightarrow & E_{max}^{(\pi, T)} & \longrightarrow & A \rtimes_{max} \Gamma \longrightarrow 0 \\
& & \downarrow = & & f_1 \downarrow & & \downarrow \lambda_{\Gamma, A} \\
0 & \longrightarrow & \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma & \longrightarrow & E_{red, max}^{(\pi, T)} & \longrightarrow & A \rtimes_{red} \Gamma \longrightarrow 0 \\
& & \downarrow \lambda_{\Gamma, \mathcal{K}(\mathcal{H}) \otimes SA} & & f_2 \downarrow & & \downarrow = \\
0 & \longrightarrow & \mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{red} \Gamma & \longrightarrow & E_{red}^{(\pi, T)} & \longrightarrow & A \rtimes_{red} \Gamma \longrightarrow 0
\end{array}$$

which satisfy the conditions of remark 3.5 relatively to s_{red} , s_{max} and $s_{red, max}$, and hence we deduce

$$(9) \quad \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma, E_{red, max}^{(\pi, T)}} \circ \lambda_{A, \Gamma, *} = \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma, E_{max}^{(\pi, T)}}$$

and

$$(10) \quad \lambda_{\mathcal{K}(\mathcal{H}) \otimes SA, \Gamma, *} \circ \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{max} \Gamma, E_{red, max}^{(\pi, T)}} = \mathcal{D}_{\mathcal{K}(\mathcal{H}) \otimes SA \rtimes_{red} \Gamma, E_{red}^{(\pi, T)}}$$

Let us set then

$$\mathcal{D}'_A = \mathcal{M}_{SA \rtimes_{max} \Gamma}^{-1} \circ \mathcal{D}_{SA \rtimes_{max} \Gamma, E_{red, max}^{(\pi, T)}} : \mathcal{K}_*(A \rtimes_{red} \Gamma) \rightarrow \mathcal{K}_*(SA \rtimes_{max} \Gamma).$$

Since we have by definition of the quantitative Kasparov transformation the equalities

$$\mathcal{J}_{\Gamma, red}([\partial_A]) = \mathcal{M}_{SA \rtimes_{red} \Gamma}^{-1} \circ \mathcal{D}_{SA \rtimes_{red} \Gamma, E_{red}^{(\pi, T)}}$$

and

$$\mathcal{J}_{\Gamma, max}([\partial_A]) = \mathcal{M}_{SA \rtimes_{max} \Gamma}^{-1} \circ \mathcal{D}_{SA \rtimes_{max} \Gamma, E_{max}^{(\pi, T)}},$$

we deduce by using equations (9) and (10), theorems 5.3, 5.5, 5.4 and 5.6 and naturality of Morita equivalence, that there exists a control pair (λ, h) such that $\mathcal{J}_{\Gamma, max}([\partial_A]^{-1}) \circ \mathcal{D}'_A$ is a (α, h) -inverse for $\lambda_{\Gamma, A, *}$. \square

6. THE QUANTITATIVE BAUM-CONNES CONJECTURE

In this section, we formulate a quantitative version for the Baum-Connes conjecture and we prove it for a large class of groups.

6.1. The Rips complex. Let Γ be a finitely generated group equipped with a length ℓ arising from a finite and symmetric generating set. Recall that for any positive number d , then the d -Rips complex $P_d(\Gamma)$ is the set of finitely supported probability measures on Γ with support of diameter less than d for the distance induced by ℓ . We equip $P_d(\Gamma)$ with the distance induced by the norm $\|h\| = \sup\{\|h(\gamma)\|; \gamma \in \Gamma\}$ for $h \in C_0(\Gamma, \mathbb{C})$. Since ℓ is a proper function, i.e. $B(e, r)$ is finite for every positive number r , we see that $P_d(\Gamma)$ is a finite dimension and locally finite simplicial complex and the action of Γ by left translations is simplicial, proper and cocompact. Let us denote by

- $V_d(\Gamma)$ the closed subset of elements of $P_d(\Gamma)$ with support in $B(e, d)$.
- $W_d(\Gamma)$ the closed subset of elements of $P_d(\Gamma)$ with support in $B(e, 2d)$;

Then $V_d(\Gamma)$ is a compact subset of $W_d(\Gamma)$ and contains a fundamental domain for the action of Γ on $P_d(\Gamma)$.

Lemma 6.1. *The compact $V_d(\Gamma)$ is contained in the interior of $W_d(\Gamma)$.*

Proof. Let h be an element in $V_d(\Gamma)$ and choose an element γ in $B(e, d)$ such that $h(\gamma) > 0$. Then if g is an element of $P_d(\Gamma)$ such that $\|g - h\| < h(\gamma)$, we get that $g(\gamma) \neq 0$ and thus every element γ' of the support of g satisfies $\ell(\gamma^{-1}\gamma') < d$. Hence g belongs to $W_d(\Gamma)$. \square

Lemma 6.2. *There is a continuous function $\phi : P_d(\Gamma) \rightarrow [0, 1]$ compactly supported in $W_d(\Gamma)$ such that*

$$\sum_{\gamma \in \Gamma} \gamma(\phi) = 1.$$

Proof. Let $\psi : P_d(\Gamma) \rightarrow [0, 1]$ a continuous function compactly supported in the interior of $W_d(\Gamma)$ and such that $\psi(x) = 1$ if x belongs to $V_d(\Gamma)$. Since $V_d(\Gamma)$ contains a fundamental domain for the action of Γ on $P_d(\Gamma)$, we get that $\sum_{\gamma \in \Gamma} \psi(\gamma x) > 0$ for all x in $P_d(\Gamma)$ (notice that the sum $\sum_{\gamma \in \Gamma} \psi(\gamma x)$ is locally finite). We define then $\phi(x) = \frac{\psi(x)}{\sum_{\gamma \in \Gamma} \psi(\gamma x)}$ for any x in $P_d(\Gamma)$. \square

Let us define $s_{\Gamma, d}$ as the cardinality of the finite set

$$\{\gamma \in \Gamma \text{ such that } \gamma W_d(\Gamma) \cap W_d(\Gamma) \neq \emptyset\}.$$

Then for any function ϕ as in lemma 6.2, the function

$$e_\phi : \Gamma \rightarrow C_0(P_d(\Gamma)); \gamma \mapsto \sum_{\gamma \in \Gamma} \phi^{1/2} \gamma(\phi^{1/2})$$

is a projection of $C_0(P_d(\Gamma)) \rtimes_{red} \Gamma$ with propagation less than $s_{\Gamma, d}$. Moreover, since the set of function satisfying the condition of lemma 6.2 is an affine space, we get that for any positive number ε and r with $\varepsilon < 1/4$ and $r \geq s_{\Gamma, d}$, the class

$$[e_\phi, 0]_{\varepsilon, r} \in K_0^{\varepsilon, r}(C_0(P_d(\Gamma)) \rtimes_{red} \Gamma)$$

does not depend on the chosen function ϕ . Let us set then $r_{\Gamma, d, \varepsilon} = k_{\mathcal{J}, \varepsilon / \alpha_J} s_{\Gamma, d}$. Recall that $k_{\mathcal{J}}$ can be chosen non increasing and in this case, $r_{\Gamma, d, \varepsilon}$ is non decreasing in d and non increasing in ε .

Definition 6.3. *For any Γ - C^* -algebra A and any positive numbers ε , r and d with $\varepsilon < 1/4$ and $r \geq r_{\Gamma, d, \varepsilon}$, we define the quantitative assembly map*

$$\begin{aligned} \mu_{\Gamma, A, *}^{\varepsilon, r, d} : KK_*^\Gamma(C_0(P_d(\Gamma)), A) &\rightarrow K_*^{\varepsilon, r}(A \rtimes_{red} \Gamma) \\ z &\mapsto \left(J_\Gamma^{red, \frac{\varepsilon}{\alpha_J}, \frac{r}{k_{\mathcal{J}, \varepsilon / \alpha_J}}} (z) \right) \left([e_\phi, 0]_{\frac{\varepsilon}{\alpha_J}, \frac{r}{k_{\mathcal{J}, \varepsilon / \alpha_J}}} \right). \end{aligned}$$

Then according to theorem 5.3, the map $\mu_{\Gamma, A}^{\varepsilon, r, d}$ is a homomorphism of groups (resp. semi-groups) in even (resp. odd) degree. For any positive numbers d and d' such that $d \leq d'$, we denote by $q_{d, d'} : C_0(P_{d'}(\Gamma)) \rightarrow C_0(P_d(\Gamma))$ the homomorphism induced by the restriction from $P_{d'}(\Gamma)$ to $P_d(\Gamma)$. It is straightforward to check that if d , d' and r are positive numbers such that $d \leq d'$ and $r \geq r_{\Gamma, d', \varepsilon}$, then $\mu_{\Gamma, A}^{\varepsilon, r, d} = \mu_{\Gamma, A}^{\varepsilon, r, d'} \circ q_{d, d', *}$. Moreover, for every positive numbers ε , ε' , d , r and r' such that $\varepsilon \leq \varepsilon' \leq 1/4$, $r_{\Gamma, d, \varepsilon} \leq r$, $r_{\Gamma, d, \varepsilon'} \leq r'$, and $r < r'$, we get by definition of a controlled morphism that

$$(11) \quad \iota_*^{\varepsilon, \varepsilon', r, r'} \circ \mu_{\Gamma, A, *}^{\varepsilon, r, d} = \mu_{\Gamma, A, *}^{\varepsilon', r', d}.$$

Furthermore, the quantitative assembly maps are natural in the Γ - C^* -algebra, i.e. if A and B are Γ - C^* -algebras and if $\phi : A \rightarrow B$ is a Γ -equivariant homomorphism, then

$$\phi_{\Gamma, red, *, \varepsilon, r} \circ \mu_{\Gamma, A, *}^{\varepsilon, r, d} = \mu_{\Gamma, B, *}^{\varepsilon, r, d} \circ \phi_*$$

for every positive numbers r and ε with $r \geq r_{\Gamma, d, \varepsilon}$ and $\varepsilon < 1/4$. These quantitative assembly maps are related to the usual assembly maps in the following way: recall from [2] that there is a bunch of assembly maps with coefficients in a Γ - C^* -algebra A defined by

$$\begin{aligned} \mu_{\Gamma, A, *}^d : KK_*^\Gamma(C_0(P_d(\Gamma)), A) &\rightarrow K_*(A \rtimes_{red} \Gamma) \\ z &\mapsto [e_\phi] \otimes_{C_0(P_d(\Gamma)) \rtimes \Gamma} J_\Gamma(z). \end{aligned}$$

For every positive numbers r and ε with $r \geq r_{\Gamma, d, \varepsilon}$ and $\varepsilon < 1/4$, we have

$$(12) \quad \iota_*^{\varepsilon, r} \circ \mu_{\Gamma, A, *}^{\varepsilon, r, d} = \mu_{\Gamma, A, *}^d.$$

Recall that since $\mu_{\Gamma, A, *}^{d'} \circ q_{d, d', *} = \mu_{\Gamma, A, *}^d$ for all positive numbers d and d' with $d \leq d'$, the family of assembly maps $(\mu_{\Gamma, A, *}^d)_{d>0}$ gives rise to a homomorphism

$$\mu_{\Gamma, A, *} : \lim_{d>0} KK_*^\Gamma(C_0(P_d(\Gamma)), A) \longrightarrow K_*(A \rtimes_{red} \Gamma)$$

called the Baum-Connes assembly map.

6.2. Quantitative statements. Let us consider for a Γ - C^* -algebra A and positive numbers $d, d', r, r', \varepsilon$ and ε' with $d \leq d'$, $\varepsilon' \leq \varepsilon < 1/4$, $r_{\Gamma, d, \varepsilon} \leq r$ and $r' \leq r$ the following statements:

$QI_{\Gamma, A, *}(d, d', r, \varepsilon)$: for any element x in $KK_*^\Gamma(C_0(P_d(\Gamma)), A)$, then $\mu_{\Gamma, A, *}^{\varepsilon, r, d}(x) = 0$ in $K_*^{\varepsilon, r}(A \rtimes_{red} \Gamma)$ implies that $q_{d, d', *}^*(x) = 0$ in $KK_*^\Gamma(C_0(P_{d'}(\Gamma)), A)$.

$QS_{\Gamma, A, *}(d, r, r', \varepsilon, \varepsilon')$: for every y in $K_*^{\varepsilon', r'}(A \rtimes_{red} \Gamma)$, there exists an element x in $KK_*^\Gamma(C_0(P_d(\Gamma)), A)$ such that

$$\mu_{\Gamma, A, *}^{\varepsilon, r, d}(x) = \iota_*^{\varepsilon', \varepsilon, r'}(y).$$

Using equation (12) and remark 1.18 we get

Proposition 6.4. *Assume that for all positive number d there exists a positive number ε with $\varepsilon < 1/4$ for which the following holds:*

*for any positive number r with $r \geq r_{\Gamma, d, \varepsilon}$, there exists a positive number d' with $d' \geq d$ such that $QI_{\Gamma, A, *}(d, d', r, \varepsilon)$ is satisfied.*

*Then $\mu_{\Gamma, A, *}$ is one-to-one.*

We can also easily prove the following:

Proposition 6.5. *Assume that there exists a positive number ε' with $\varepsilon' < 1/4$ such that the following holds:*

*for any positive number r' , there exist positive numbers ε, d and r with $\varepsilon' \leq \varepsilon < 1/4$, $r_{\Gamma, d, \varepsilon} \leq r$ and $r' \leq r$ such that $QS_{\Gamma, A, *}(d, r, r', \varepsilon, \varepsilon')$ is true.*

*Then $\mu_{\Gamma, A, *}$ is onto.*

The following results provide numerous examples of finitely generated groups that satisfy the quantitative statements.

Theorem 6.6. *Let A be a Γ - C^* -algebra. Then the following assertions are equivalent:*

- (i) $\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}$ is one-to-one,
- (ii) For any positive numbers d, ε and $r \geq r_{\Gamma, d, \varepsilon}$ with $\varepsilon < 1/4$ and $r \geq r_{\Gamma, d}$, there exists a positive number d' with $d' \geq d$ for which $QI_{\Gamma, A}(d, d', r, \varepsilon)$ is satisfied.

Proof. Assume that condition (ii) holds.

Let x be an element in some $KK_*^\Gamma(C_0(P_d(\Gamma)), \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A))$ such that

$$\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}^d(x) = 0.$$

Using equation (12), we get that $\iota_*^{\varepsilon', r'}(\mu_{\Gamma, A, *}^{\varepsilon', r', d}(x)) = 0$ for any ε' in $(0, 1/4)$ and $r' \geq r_{\Gamma, d, \varepsilon'}$ and hence, by remark 1.18, we can find ε and $r > r_{\Gamma, d, \varepsilon}$ such that $\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}^{\varepsilon, r, d}(x) = 0$. Recall from [14, Proposition 3.4] that we have an isomorphism

$$(13) \quad KK_0^\Gamma(C_0(P_d(\Gamma)), \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A)) \xrightarrow{\cong} KK_0^\Gamma(C_0(P_d(\Gamma)), A)^\mathbb{N}$$

induced on the j th factor and up to the Morita equivalence

$$KK_0^\Gamma(C_0(P_d(\Gamma)), A) \cong KK_0^\Gamma(C_0(P_d(\Gamma)), \mathcal{K}(\mathcal{H}) \otimes A)$$

by the j th projection $\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A) \rightarrow \mathcal{K}(\mathcal{H}) \otimes A$. Let $(x_i)_{i \in \mathbb{N}}$ be the element of $KK_0^\Gamma(C_0(P_d(\Gamma)), A)^\mathbb{N}$ corresponding to x under this identification and let $d' \geq d$ be a number such that $QI_{\Gamma, A}(d, d', r, \varepsilon)$ holds. Naturality of the quantitative assembly maps implies that $\mu_{\Gamma, A, *}^{\varepsilon, r, d}(x_i) = 0$ and hence that $q_{d, d', *}(x_i) = 0$ in $KK_*^\Gamma(C_0(P_{d'}(\Gamma)), A)$ for every integer i . Using once again the isomorphism of equation (13), we get that $q_{d, d', *}(x) = 0$ in $KK_*^\Gamma(C_0(P_{d'}(\Gamma)), \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A))$ and hence $\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}^{\varepsilon, r, d}$ is one-to-one.

Let us prove the converse in the even case, the odd case being similar. Assume that there exists positive numbers d, ε and r with $\varepsilon < 1/4$ and $r \geq r_{\Gamma, d, \varepsilon}$ and such that for all $d' \geq d$, the condition $QI_{\Gamma, A}(d, d', r, \varepsilon)$ does not hold. Let us prove that $\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}$ is not one-to-one. Let $(d_i)_{i \in \mathbb{N}}$ be an increasing and unbounded sequence of positive numbers such that $d_i \geq d$ for all integer i . For all integer i , let x_i be an element in $KK_0^\Gamma(C_0(P_d(\Gamma)), A)$ such that $\mu_{\Gamma, A, *}^{\varepsilon, r, d}(x_i) = 0$ in $K_0(A \rtimes_{red} \Gamma)$ and $q_{d, d_i, *}(x_i) \neq 0$ in $KK_0^\Gamma(C_0(P_{d_i}(\Gamma)), A)$. Let x be the element of $KK_0^\Gamma(C_0(P_d(\Gamma)), \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A))$ corresponding to $(x_i)_{i \in \mathbb{N}}$ under the identification of equation (13). Let $(p_i)_{i \in \mathbb{N}}$ be a family of ε - r -projections, with p_i in some $M_i(A \rtimes_{red} \Gamma)$ and n an integer such that

$$\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}^{\varepsilon, r, d}(x) = [(p_i)_{i \in \mathbb{N}}, n]_{\varepsilon, r}$$

in $K_0^{\varepsilon, r}(\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A) \rtimes_{red} \Gamma)$. By naturality of $\mu_{\Gamma, \bullet, *}^{\varepsilon, r, d}$, we get that $[p_i, n]_{\varepsilon, r} = 0$ in $K_0^{\varepsilon, r}(A \rtimes_{red} \Gamma)$ for all integer i . We see by using proposition 1.31 that then $\iota_*^{\varepsilon, r}([(p_i)_{i \in \mathbb{N}}, n]) = 0$ in $K_0(\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A) \rtimes_{red} \Gamma)$. We eventually obtain that $\mu_{\Gamma, A}^d(x) = \iota_*^{\varepsilon, r} \circ \mu_{\Gamma, A}^{\varepsilon, r, d}(x) = 0$. Since $q_{d, d_i, *}(x) \neq 0$ for every integer i , we get that $\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}^{\varepsilon, r, d}$ is not one-to-one. \square

Theorem 6.7. *There exists $\lambda > 1$ such that for any Γ - C^* -algebra, the following assertions are equivalent:*

- (i) $\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}$ is onto;

- (ii) For any positive numbers ε and r' with $\varepsilon < \frac{1}{4\lambda}$, there exist positive numbers d and r with $r_{\Gamma,d,\varepsilon} \leq r$ and $r' \leq r$ for which $QS_{\Gamma,A}(d, r, r', \lambda\varepsilon, \varepsilon)$ is satisfied.

Proof. Choose λ as in remark 1.18. Assume that condition (ii) holds. Let z be an element in $K_*(\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A) \rtimes_{red} \Gamma)$ and let y be an element in $K_*^{\varepsilon, r'}(\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A) \rtimes_{red} \Gamma)$ such that $\iota_*^{\varepsilon, r'}(y) = z$, with $0 < \varepsilon < \frac{1}{4\lambda}$ and $r' > 0$. Let y_i be the image of y under the composition

$$(14) \quad K_*^{\varepsilon, r'}(\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A) \rtimes_{red} \Gamma) \rightarrow K_*^{\varepsilon, r'}(\mathcal{K}(\mathcal{H}) \otimes A \rtimes_{red} \Gamma) \xrightarrow{\cong} K_*^{\varepsilon, r'}(A \rtimes_{red} \Gamma),$$

where the first map is induced by the evaluation $\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A) \rightarrow \mathcal{K}(\mathcal{H}) \otimes A$ at i and the second map is the Morita equivalence of proposition 1.29. Let d and r be numbers with $r \geq r'$ and $r \geq r_{\Gamma,d,\varepsilon}$ and such that $QS_{\Gamma,A}(d, r, r', \lambda\varepsilon, \varepsilon)$ holds. Then for any integer i , there exists a x_i in $KK_*^\Gamma(C_0(P_d(\Gamma)), A)$ such that $\mu_{\Gamma,A,*}^{\lambda\varepsilon, r, d}(x_i) = \iota_*^{\varepsilon, \lambda\varepsilon, r', r}(y_i)$ in $K_*^{\varepsilon, r}(A \rtimes_{red} \Gamma)$. Let

$$x \in KK_*^\Gamma(C_0(P_d(\Gamma)), \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A))$$

be the element corresponding to $(x_i)_{i \in \mathbb{N}}$ under the identification of equation (13). By naturality of the quantitative assembly maps, we get according to proposition 1.31 and up to replace λ by 3λ (for the odd case) that

$$\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}^{\lambda\varepsilon, r, d}(x) = \iota_*^{\varepsilon, \lambda\varepsilon, r', r}(y)$$

in $K_*^{\varepsilon, r}(\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A) \rtimes_{red} \Gamma)$. We have hence

$$\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}^d(x) = \iota_*^{\varepsilon, r'}(y) = z,$$

and therefore $\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}$ is onto.

Let us prove the converse in the even case, the odd case being similar. Assume that there exist positive numbers ε and r' with $\varepsilon < \frac{1}{4\lambda}$ such that for all positive numbers r and d with $r \geq r'$ and $r \geq r_{\Gamma,d,\varepsilon}$, then $QS_{\Gamma,A}(d, r, r', \lambda\varepsilon, \varepsilon)$ does not hold. Let us prove then that $\mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}$ is not onto. Let $(d_i)_{i \in \mathbb{N}}$ and $(r_i)_{i \in \mathbb{N}}$ be increasing and unbounded sequences of positive numbers such that $r_i \geq r_{\Gamma,d_i,\lambda\varepsilon}$ and $r_i \geq r'$. Let y_i be an element in $K_0^{\varepsilon, r'}(A \rtimes_{red} \Gamma)$ such that $\iota_*^{\varepsilon, \lambda\varepsilon, r', r_i}(y_i)$ is not in the range of $\mu_{\Gamma,A,*}^{\lambda\varepsilon, r_i, d_i}$. There exists an element y in $K_0^{\varepsilon, r'}(\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A) \rtimes_{red} \Gamma)$ such that for every integer i , the image of y under the composition of equation (14) is y_i . Assume that for some d' , there is an x in $KK_0^\Gamma(C_0(P_{d'}(\Gamma)), \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A))$ such that $\iota_*^{\varepsilon, r'}(y) = \mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}^{d'}(x)$. Using remark 1.18, we see that there exists a positive number r with $r' \leq r$ and $r_{\Gamma,d',\lambda\varepsilon} \leq r$ and such that

$$\iota_*^{\varepsilon, \lambda\varepsilon, r', r} \circ \mu_{\Gamma, \ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A), *}^{\varepsilon, r', d'}(x) = \iota_*^{\varepsilon, \lambda\varepsilon, r', r}(y).$$

But then, if we choose i such that $r_i \geq r$ and $d_i \geq d'$ we get by using naturality of the assembly map and equation (11) that $\iota_*^{\varepsilon, \lambda\varepsilon, r', r_i}(y_i)$ belongs to the image of $\mu_{\Gamma,A,*}^{\lambda\varepsilon, r_i, d_i}$, which contradicts our assumption. \square

Replacing in the proof of (ii) implies (i) of theorems 6.6 and 6.7 the algebra $\ell^\infty(\mathbb{N}, \mathcal{K}(\mathcal{H}) \otimes A)$ by $\prod_{i \in \mathbb{N}} (\mathcal{K}(\mathcal{H}) \otimes A_i)$ for a family $(A_i)_{i \in \mathbb{N}}$ of Γ - C^* -algebras, we can prove the following result.

Theorem 6.8. *Let Γ be a discrete group.*

- (i) *Assume that for any Γ - C^* -algebra A , the assembly map $\mu_{\Gamma,A,*}$ is one-to-one. Then for any positive numbers d, ε and $r \geq r_{\Gamma,d,\varepsilon}$ with $\varepsilon < 1/4$ and $r \geq r_{\Gamma,d}$, there exists a positive number d' with $d' \geq d$ such that $QI_{\Gamma,A}(d, d', r, \varepsilon)$ is satisfied for every Γ - C^* -algebra A ;*
- (ii) *Assume that for any Γ - C^* -algebra A , the assembly map $\mu_{\Gamma,A,*}$ is onto. Then for some $\lambda > 1$ and for any positive numbers ε and r' with $\varepsilon < \frac{1}{4\lambda}$, there exist positive numbers d and r with $r_{\Gamma,d,\varepsilon} \leq r$ and $r' \leq r$ such that $QS_{\Gamma,A}(d, r, r', \lambda\varepsilon, \varepsilon)$ is satisfied for every Γ - C^* -algebra A .*

In particular, if Γ satisfies the Baum-Connes conjecture with coefficients, then Γ satisfies points (i) and (ii) above.

Recall from [16, 20] that if Γ coarsely embeds in a Hilbert space, then $\mu_{\Gamma,A,*}$ is one-to-one for every Γ - C^* -algebra A . Hence we get:

Corollary 6.9. *If Γ coarsely embeds in a Hilbert space, then for any positive numbers d, ε and $r \geq r_{\Gamma,d,\varepsilon}$ with $\varepsilon < 1/4$ and $r \geq r_{\Gamma,d}$, there exists a positive number d' with $d' \geq d$ such that $QI_{\Gamma,A}(d, d', r, \varepsilon)$ is satisfied for every Γ - C^* -algebra A ;*

The quantitative assembly maps admit maximal versions defined with notations of definition 6.3 for any Γ - C^* -algebra A and any positive number ε, r and d with $\varepsilon < 1/4$ and $r \geq r_{\Gamma,d,\varepsilon}$, as

$$\begin{aligned} \mu_{\Gamma,A,max,*}^{\varepsilon,r,d} : KK_*^\Gamma(C_0(P_d(\Gamma)), A) &\rightarrow K_*^{\varepsilon,r}(A \rtimes_{max} \Gamma) \\ z &\mapsto (J_\Gamma^{max, \frac{\varepsilon}{\alpha_J}, \frac{r}{K_{J,\varepsilon/\alpha_J}}}(z)) \left([e_\phi, 0]_{\frac{\varepsilon}{\alpha_J}, \frac{r}{K_{J,\varepsilon/\alpha_J}}} \right). \end{aligned}$$

As in the reduced case, we have using the same notations

- for any positive number d and d' such that $d \leq d'$, then

$$\mu_{\Gamma,A,max,*}^{\varepsilon,r,d} = \mu_{\Gamma,A,max,*}^{\varepsilon,r,d'} \circ q_{d,d',*}.$$

- for every positive numbers $\varepsilon, \varepsilon', d, r$ and r' such that $\varepsilon \leq \varepsilon' \leq 1/4$, $r_{\Gamma,d,\varepsilon} \leq r$, $r_{\Gamma,d,\varepsilon'} \leq r'$, and $r < r'$, then

$$\iota_*^{\varepsilon,\varepsilon',r,r'} \circ \mu_{\Gamma,A,max,*}^{\varepsilon,r,d} = \mu_{\Gamma,A,max,*}^{\varepsilon',r',d}.$$

- the maximal quantitative assembly maps are natural in the Γ - C^* -algebras.

Moreover, by theorem 5.5(i), the maximal quantitative assembly maps are compatible with the reduced ones, i.e $\mu_{\Gamma,A,*}^{\varepsilon,r,d} = \lambda_{\Gamma,A,*}^{\varepsilon,r} \circ \mu_{\Gamma,A,max,*}^{\varepsilon,r,d}$. The surjectivity of the Baum-Connes assembly map $\mu_{\Gamma,A,*}$ implies that the map

$$\lambda_{\Gamma,A,*} : K_*(A \rtimes_{max} \Gamma) \rightarrow K_*(A \rtimes_{red} \Gamma)$$

is onto. We have a similar statement in the setting of quantitative K -theory.

Theorem 6.10. *There exists $\lambda > 1$ such the following holds : let Γ be a discrete group and assume that for any Γ - C^* -algebra A , the assembly map $\mu_{\Gamma,A,*}$ is onto. Then for any positive numbers ε and r , with $\varepsilon < \frac{1}{4\lambda}$, there exists a positive number r' with $r' \geq r$ such that*

- *for any Γ - C^* -algebra A ;*
- *for any x in $K_*^{\varepsilon,r}(A \rtimes_{red} \Gamma)$,*

there exists y in $K_^{\lambda\varepsilon,r'}(A \rtimes_{max} \Gamma)$ such that $\lambda_{\Gamma,A,*}^{\lambda\varepsilon,r'}(y) = \iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(x)$.*

7. FURTHER COMMENTS

The definition of quantitative K -theory can be extended to the framework of filtered Banach algebras, i.e. Banach algebra A equipped with a family $(A_r)_{r>0}$ of linear subspaces indexed by positive numbers such that:

- $A_r \subset A_{r'}$ if $r \leq r'$;
- $A_r \cdot A_{r'} \subset A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in A .

Since we no more have an involution, we need to introduce instead a norm control for almost idempotents. Let ε be in $(0, 1/4)$ and let r and N be positive numbers. An element e of A is an ε - r - N -idempotent if

- e is in A_r ;
- $\|e^2 - e\| < \varepsilon$;
- $\|e\| < N$;

Similarly, if A is a unital, an element x in A is called ε - r - N -invertible if

- x is in A_r ;
- $\|x\| < N$;
- there exists an element y in A_r such that $\|y\| < N$, $\|xy - 1\| < \varepsilon$ and $\|yx - 1\| < \varepsilon$.

Quantitative K -theory can then be defined in the setting of ε - r - N -idempotents and of ε - r - N -invertibles. We obtain in this way a bunch of abelian semi-groups $(K_*^{\varepsilon, r, N}(A))_{\varepsilon \in (0, 1/4), r > 0, N > 1}$. Let us set for a fixed $N > 1$

$$\mathcal{K}_*^N(A) = (K_*^{\varepsilon, r, N}(A))_{\varepsilon \in (0, 1/4), r > 0}.$$

If A is a filtered C^* -algebra and e an ε - r - N -idempotent in A , then there is an obvious $(1, 1)$ -controlled morphism $\mathcal{K}_0(A) \rightarrow \mathcal{K}_0^N(A)$. Approximating $((2e^* - 1)(2e - 1) + 1)^{1/2}e((2e^* - 1)(e - 1) + 1)^{-1/2}$ by using a power series (compare with the proof of lemma 1.10), we get that for every $N > 1$, there exists a control pair (λ_N, h_N) such that $\mathcal{K}_0(A) \rightarrow \mathcal{K}_0^N(A)$ is a (λ_N, h_N) -controlled isomorphism. Using the polar decomposition, we have a similar statement in the odd case.

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